2.1 Higher Order ODE

· A linear ODE of order n is given by

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots \quad a_1(x)\frac{dy}{dx} + a_0(x)y = f(x) \cdots 2.1.1$$

If $f(x)=0$ the eqn is called homogeneous, otherwise it is inhomogeneous. First order linear equation is a special case of (2.1.1.) The general solution will contain n arbitrary constants which can be determined if n boundary conditions are given. To solve eqn (2.1.1) the general solution of the complementary eq^n i.e. 2.1.1 with $f(x)=0$ must be obtained.

i.e. one has to find general solution

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

$$\cdots (2.1.2)$$

We are to find n linearly independent function that satisfy eq. 2.1.2.. The general solution is

$$y_c(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots \quad c_n y_n(x) \cdots 2.1.3.$$

where c_n's are arbitrary constants to be determined from boundary conditions. Here $y_c(x)$ is called the complementary function of 2.1.1.

Note For n functions to be linearly independent the relation

$$c_1 y_1(x) + c_2 y_2(x) + \cdots \quad c_n y_n(x) = 0 \cdots 2.1.4$$

should hold for all $c_1 = c_2 = c_3 = \cdots = 0$

if $\sum_i c_i y_i(x) = 0$ in which not all c_i's are zero, then the functions are linearly dependent.

If

$$\sum c_i y_i(x) = 0$$

then

$$\sum_i c_i y_i'(x) = 0$$

$$\sum_i c_i y_i''(x) = 0 \qquad 2.1.5$$

$$\cdots\cdots$$

$$\sum c_i y_i^{(n-1)}(x) = 0$$

where $y_i^{n-1}(x) = \dfrac{d^{n-1}y}{dx^{n-1}}$ etc.

From the method of solutions for simultaneous linear equations in n variables we know that if the determinant of the coefficient is non zero, then all c's are zero. Then $y_1(x), y_2(x), \ldots$ etc are linearly independent. Non trivial solutions exist when the det is zero. But this does not guarantee that y's are linearly dependent.

If $f(x) = 0$ in eq (2.1.1) then y_c is the general solution. If $f(x) \neq 0$ then y_c is part of the solution and the general solution is

$$y = y_c + y_p$$

Where y_p is the particular integral which can be any function satisfying 2.1.1 provided it is linearly independent of y_c.

2.2. 2nd Order Linear Differential Equations :-

These type of equations have been ~~extree~~ extensively studied and have many applications in physical science. The general linear second order DE can be written as

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x) \quad \ldots\ldots \qquad 2.2.1$$

Here $P(x)$, $Q(x)$ and $R(x)$ are functions of x alone or may be constants.

In general 2.2.1 can not be solved explicitely in terms of known elementary functions and most of the times the solution is obtained using infinite series. The overall ideas for solving (2.2.1) may be applied for higher order DE's. Let us first see if at all there exists a solution of 2.2.1.

Theorem A:

Let $P(x)$, $Q(x)$, $R(x)$ be continuous functions in a closed interval $[a,b]$ {i.e. $a \leq x \leq b$} [If the interval is open $a<x<b$]. If x_0 is any point in $[a,b]$ and if y_0 and y_0' are any numbers whatsoever, then eqn 2.2.1 has one and only one solution $y(x)$ on the interval such that $y(x_0) = y_0$ and $y'(x_0) = y_0'$.

Under this ~~hy~~ hypothesis at any point x_0 in $[a,b]$ we can arbitrarily prescribe the values of $y(x)$ and $y'(x)$ and there will then exist precisely one solution of (2.2.1) on $[a,b]$ that assumes the prescribed values at the given point. More generally (2.2.1) has a unique solution on $[a,b]$ that passes through a specified point (x_0, y_0) with a specified slope y_0'. If $R(x)$ is zero then we get the homogeneous eqn

$$y'' + P(x)y' + Q(x)y = 0 \quad \ldots \quad 2.2.2$$

For $R(x) \neq 0$ the eq^n 2.2.1 is inhomogeneous or nonhomogeneous.

For a nonhomogeneous eq^n it is necessary to consider the homogeneous eq^n first.

Suppose $y_g(x, c_1, c_2)$ is the general solution of 2.2.2 and $y_p(x)$ is a fixed particular solution of 2.2.1.

Then $y(x) - y_p(x)$ is a sol^n of 2.2.2.

Proof: $(y'' - y_p'') + P(x)(y' - y_p') + Q(x)(y - y_p)$

$$= \{y'' + P(x)y' + Q(x)y\} - \{y_p''(x) + P(x)y_p'(x) + Q(x)y_p(x)\}$$

$$= R(x) - R(x) = 0$$

since $y_g(x, c_1, c_2)$ is the general solution of (2.2.2)

$$y_g(x, c_1, c_2) = y(x) - y_p(x)$$

or $y(x) = y_g(x, c_1, c_2) + y_p(x) \quad \ldots \quad (2.2.3)$

Theorem B:-

Thus we have the theorem if y_g is the general solution of 2.2.2 and y_p is any particular solution of (2.2.1) then $y_g + y_p$ is the general solution of 2.2.1.

We will see that if y_g is known then a formal procedure is available for finding y_p. Thus the central problem is to solve the homogeneous eqn 2.2.2.

A trivial solution of eq. 2.2.2 is $y(x)=0$ for $\in x$. This is of no interest.

The following theorem shows the basic structural fact about solutions of eq. 2.2.2.

Theorem C:

If $y_1(x)$ and $y_2(x)$ are any solutions of eq. 2.2.2 then $c_1 y_1(x) + c_2 y_2(x)$ is also a solution for any constants c_1 & c_2

Proof: This follows immediately as

$$(c_1 y_1'' + c_2 y_2'') + P(x)(c_1 y_1' + c_2 y_2') + Q(x)(c_1 y_1 + c_2 y_2)$$

$$= c_1\left[y_1'' + P(x)y_1' + Q(x)y_1\right] + c_2\left[y_2'' + P(x)y_1' + Q(x)y_1\right]$$

$$= 0$$

Thus any linear combination of the two solutions of the homogeneous eqn is also a solution.

If $y_2 = \lambda y_1$ say, then there is only one constant. Thus if neither y_1 nor y_2 is a constant multiple of the other, then

$$c_1 y_1(x) + c_2 y_2(x)$$

will be the general solution of eq. 2.2.2.

All these properties follow from the linearity of the differential operator.

2.3. General solution of the homogeneous Eq^n:

Let $y_1(x)$ and $y_2(x)$ be the linearly independent solutions of the homogeneous eqn. 2.2.2.

$$y'' + P(x)y' + Q(x)y = 0 \quad \ldots \quad 2.2.2$$

on the interval $[a,b]$. Then

$$c_1 y_1(x) + c_2 y_2(x) \quad \ldots \quad 2.3.1$$

is the general solution of (2.2.2) on $[a,b]$ in the sense that every solution of 2.2.2 on this interval can be obtained from 2.3.1 by a suitable choice of c_1 and c_2.

Proof:

Let $y(x)$ be a solution of (2.2.2) on $[a,b]$. We are to show that constants c_1 and c_2 can be found such that

$$y(x) = c_1 y_1(x) + c_2 y_2(x) \text{ for all } x \text{ in } [a,b]$$

We know from previous theorem that a solution of 2.2.2 over all of $[a,b]$ is completely determined by its value and that of the derivative at a single point. Since $c_1 y_1(x) + c_2 y_2(x)$ and $y(x)$ are solutions of 2.2.2 on $[a,b]$, we are to show that for some pt. (x_0) in $[a,b]$ we can find c_1 and c_2 so that

$$c_1 y_1(x_0) + c_2 y_2(x_0) = y(x_0)$$

and $c_1 y_1'(x_0) + c_2 y_2'(x_0) = y'(x_0)$

To have no trivial c_1 & c_2, the det $\begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} \neq 0$

This is called the Wronskian W of y_1 and y_2 at x_0

$\therefore$ In general $W = W(y_1, y_2) = y_1 y_2' - y_2 y_1'$ and the location x_0 is of not importance or consequence.

Lemma 1:

If $y_1(x)$ and $y_2(x)$ are any two solutions of eq. 2.2.2 on $[a,b]$ then their Wronskian $W(y_1, y_2)$ is either identically zero or never zero.

Proof: $W(y_1, y_2) = y_1 y_2' - y_1' y_2$

$$\therefore W' = \cancel{y_1' y_2'} + y_1 y_2'' - y_1'' y_2 - \cancel{y_1' y_2'}$$

$$= y_1 y_2'' - y_1'' y_2$$

Since y_1 and y_2 are both solutions of 2.2.2

$$y_1'' + P(x) y_1' + Q(x) y_1 = 0$$
$$y_2'' + P(x) y_2' + Q(x) y_2 = 0$$

Multiply 1st by y_2 and 2nd by y_1 and subtracting

$$y_1 y_2'' - y_2 y_1'' + P(x)(y_1 y_2' - y_2 y_1') + Q(x)(y_1 y_2 - y_2 y_1) = 0$$

or $y_1 y_2'' - y_2 y_1'' + P(x)(y_1 y_2' - y_2 y_1') = 0$

or $\dfrac{dW}{dx} + PW = 0$

This is a first order DE and the general solution is $W = c e^{-\int P dx}$

since the exponential factor is never zero we get the proof.

Lemma 2:

If $y_1(x)$ and $y_2(x)$ are two solutions of eq. 2.2.2 on $[a,b]$ then they are linearly dependent if and only if their Wronskian $y_1 y_2' - y_2 y_1'$ is identically zero.

Note:

If the ratio of two solutions $\frac{y_1}{y_2}$ = not constant the solutions are linearly independent

Proof:

Let y_1 and y_2 be linearly dependent

We will show that $W = y_1 y_2' - y_2 y_1' = 0$

i) If either y_1 or y_2 is zero then $W = 0$ in some $[a, b]$

(ii) We assume that y_1 or y_2 are not identically zero.

Let $y_2 = c y_1$ for some c_1

$\therefore y_2' = c y_1'$

$\therefore W = y_1 y_2' - y_2 y_1 = c(y_1 y_1' - y_1 y_1') = 0$

Let us now assume that $W = 0$ identically. We will show that $y_2 = c y_1$

let y_1 nonvanishing within $[a, b]$

Then $\frac{W}{y_1^2} = \frac{y_1 y_2' - y_2 y_1'}{y_1^2} = \left(\frac{y_2}{y_1}\right)' = 0$

$\therefore \frac{y_2}{y_1} = k$ (const)

or $y_2 = k y_1$

This shows within the interval

$y_2' = k y_1'$

Thus $y_2(x) = k\, y_1(x)$ for all x in $[a, b]$

EX: show that $y = c_1 \sin x + c_2 \cos x$ is the gen. soln of $y'' + y = 0$ on any interval and find the particular soln for which $y(0) = 2$, $y'(0) = 3$

$\sin x$ & $\cos x$ are obvious solutions; $y_1 / y_2 = \tan x \neq$ const.

Hence, are linearly independent. Here $W = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -1 \neq 0$

Particular soln $\Rightarrow$

$c_1 \sin 0 + c_2 \cos 0 = 2 \quad \therefore c_2 = 2$

$c_1 \cos 0 - c_2 \sin 0 = 3 \quad \therefore c_1 = 3$

$\therefore y(x) = 3 \sin x + 2 \cos x$ is the particular soln.

2.4. HOMOGENEOUS EQUATION WITH CONSTANT COEFFICIENTS

We have
$$y'' + P(x)y' + Q(x)y = 0 \quad \ldots \quad 2.2.2$$

Here $\left.\begin{matrix} P(x) \\ Q(x) \end{matrix}\right\}$ constants $\left\{\begin{matrix} p \\ q \end{matrix}\right.$

$$\therefore \quad y'' + py' + qy = 0 \quad \ldots \quad 2.4.1$$

Consider a sol^n
$$y = a e^{\lambda x} \quad \ldots \quad 2.4.2$$

This assumption is due to the fact that derivatives of exponentials are multiples of the $f^n s$ themselves upon differentiation

$$a(\lambda^2 + p\lambda + q)e^{\lambda x} = 0$$

or $\lambda^2 + p\lambda + q = 0 \qquad [e^{\lambda x} \neq 0] \quad \ldots \quad 2.4.3$

will be the auxilliary eq^n

$$\therefore \quad \lambda = \left[-p \pm \sqrt{p^2 - 4q}\right]/2 \quad \ldots \quad 2.4.4.$$

3 cases arise

i) Real and distinct roots

Here $p^2 > 4q$

We have $e^{\lambda_1 x}$ and $e^{\lambda_2 x}$ are solutions. Since the ratio $e^{\lambda_1 (x)}/e^{\lambda_2 x} = e^{(\lambda_1 - \lambda_2)x} \neq$ const.

the solutions are linearly independent

Thus
$$y = a_1 e^{\lambda_1 x} + a_2 e^{\lambda_2 x} \quad \ldots \quad 2.4.5$$

with λ_1 and λ_2 given by 2.4.2 is the general solution.

(ii) complex distinct roots:

Here $p^2 - 4q < 0$

Here λ_1 and λ_2 may be written as

$a \pm ib$

and by Euler's formula
$$e^{i\theta} = \cos\theta + i\sin\theta$$

The two solutions are

$$e^{\lambda_1 x} = e^{(a+ib)x} = e^{ax}\left[\cos bx + i\sin bx\right]$$
$$e^{\lambda_2 x} = e^{(a-ib)x} = e^{ax}\left[\cos bx - i\sin bx\right]$$

The general sol^n is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$ $\Rightarrow$ This consists of complex coefficients.

Since we are interested in solutions which are real valued functions we can take linear combinations of the solutions such that real valued f^{n}s are obtained.

$$\frac{1}{2}\left[e^{\lambda_1 x} + e^{\lambda_2 x}\right] = e^{ax}\cos bx$$
$$\frac{1}{2i}\left[e^{\lambda_1 x} - e^{\lambda_2 x}\right] = e^{ax}\sin bx$$

These are linearly independent and hence the general solution is

$$y(x) = e^{ax}\left[c_1\cos bx + c_2\sin bx\right] \quad \ldots\ldots \quad 2.4.6$$

This can be transformed as

$$y(x) = Ae^{ax}\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(bx+\phi) \quad \ldots\ldots \quad 2.4.7$$

<u>Note</u>: A complex valued function $w(x) = u(x) + iv(x)$ satisfies the DE in which p & q are real numbers, if and only if $u(x)$ and $v(x)$ satisfies separately eq. 2.2.2. Accordingly a complex solution of the DE always contains two real solutions and we can easily get the solution (2.4.6) from individual solutions.

(iii) Equal Real roots :-

If $\lambda_1 = \lambda_2$ then $\sqrt{p^2-4q} = 0$

we get only one solution with $\lambda = -p/2$ and $y \cong e^{\lambda x}$

we can find another linearly independent solution by inspection or by a procedure described below.

When there are repeated roots we get less number of linearly independent solutions and the Wronskian contain two or more rows or columns identical and hence vanishes. By direct substitution one finds that $xe^{\lambda x}$ is also a solⁿ of eq. 2.2.2 and in general

$xe^{\lambda x}, x^2 e^{\lambda x}, x^3 e^{\lambda x} \dots x^{k-1} e^{\lambda x}$ are solutions of general nth order ODE. Hence proper linearly independent solutions can be obtained.

In this particular case

$y = e^{\lambda x}$ and $y = xe^{\lambda x}$

are linearly independent solutions.

The general solution is

$$y = c_1 e^{\lambda x} + c_2 x e^{\lambda x}$$

with $\lambda = -p/2$

Find the complementary fⁿ of

$$y'' - 2y' + y = e^x$$

Let $y'' - 2y' + y = 0 \Rightarrow$ Homogeneous Part.

Put $y = Ae^{\lambda x}$

Then $\lambda^2 - 2\lambda + 1 = 0$

or $\lambda = 1$ (Repeated Root : double)

Although e^x is a sol^n we have to findout another solution which is linearly independent to e^x. By direct substitution we find that xe^x is such a sol^n as

$$y' = (1+x)e^x$$

$$y'' = (2+x)e^x$$

$$y'' - 2y' + y = \{2+x - 2(1+x) + x\}e^x = 0$$

Thus the complementary function is

$$y_c(x) = (c_1 + c_2 x)e^x$$

2.5. The use of a known solution to find the other!

DE

$$y'' + P(x)y' + Q(x)y = 0 \quad \cdots \quad 2.2.2$$

Two linearly independent solution

y_1 & y_2

How to find y_1 & y_2 ?

No general method exists.

However if y_1 is known by some means, y_2 can be determined by a standard procedure.

Let y_1 be a non zero solution of 2.2.2

cy_1 is also a solution $\rightarrow c = \text{Const}$.

$\Rightarrow$ Replace c by a f^n $v(x)$ such that $y_2 = v y_1$ will be a solution of 2.2.2

This works!

The linear independence of y_2 & y_1 requires that

$$\frac{y_2}{y_1} \neq \text{constant}$$
$$= v(x) \quad \text{say.}$$

[If $f(x)$ and $g(x)$ are such that $g(x) = \lambda f(x)$, then $f(x)$ and $g(x)$ are linearly dependent. If $g(x) \neq \lambda f(x)$ where λ is a constant then they are linearly independent. If $f(x) = 0$ then $f(x)$ & $g(x)$ are linearly dependent for all $g(x)$.

Assume $y_2 = v(x) y_1$ is another solution of 2.2.2

Then $\quad y_2'' + P(x) y_2' + Q(x) y_2 = 0$

Substitute $\quad y_2 = v y_1$

then $\quad y_2' = v' y_1 + v y_1'$

$$y_2'' = v'' y_1 + v' y_1' + v' y_1' + v y_1''$$

$$\therefore\ v'' y_1 + 2v' y_1' + v y_1'' + P v' y_1 + P v y_1' + Q v y_1 = 0$$

$$\therefore\ v(y_1'' + P y_1' + Q y_1) + v'' y_1 + v'(2y_1' + P y_1) = 0$$

y_1 is a sol^n of 2.2.2

$$\therefore\ v'' y_1 + v'(2y_1' + P y_1) = 0$$

$$\text{or } \frac{v''}{v'} = -\left(\frac{2y_1' + P y_1}{y_1}\right)$$
$$= -2\frac{y_1'}{y_1} - P$$

$$\int \frac{d\left(\frac{dv}{dx}\right)}{v'} = -2\int \frac{dy_1}{y_1} - \int P\,dx$$

Hence $\quad \ln v' = -2 \ln y_1 - \int P\,dx$

$$\ln(v' y_1^2) = -\int P\,dx$$

or $v' = \frac{1}{y_1^2} e^{-\int p\,dx}$

Hence $v = \int \frac{1}{y_1^2} e^{-\int p dx} dx$ 2.5.1

Thus the sol^{n}s are y_1 & vy_1

One has to show that

y_1 & vy_1 are linearly independent

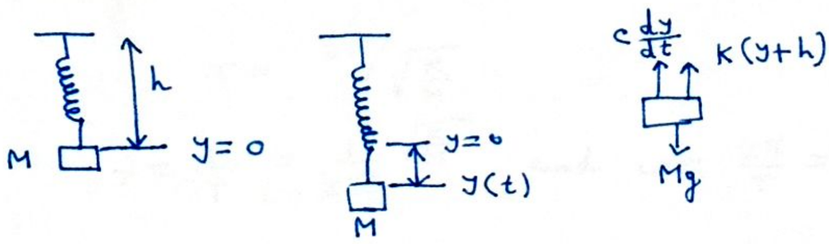
This can be done by evaluating the Wronskian, and showing that $W \neq 0$.

Ex. Oscillation of a spring with attached mass M.

Static equilibrium

$Mg = hk$

k = spring constant (Modulus)
= Force/unit displacement



Consider a resistive force ∝ velocity (viscous force)

The eqn of motion is

$$Mg - c\frac{dy}{dt} - k(y+h) = M\frac{d^2y}{dt^2}$$

$$\therefore M\frac{d^2y}{dt^2} + c\frac{dy}{dt} + \frac{k}{\cancel{M}} y = 0$$

or $$\frac{d^2y}{dt^2} + \frac{c}{M}\frac{dy}{dt} + \frac{k}{M} y = 0$$

or $y'' + py' + qy = 0$ where $p = \frac{c}{M}$ & $q = k/M$

For $p \sim 0$ [undamped motion]

$y'' + qy = 0$

Assume the solution $y = e^{\lambda t}$

$$\therefore \lambda^2 + q = 0$$

$$\lambda = \pm\sqrt{-q} = \pm i\sqrt{\frac{K}{M}} = i\sqrt{\frac{K}{M}},\ -i\sqrt{\frac{K}{M}}$$

The solution is

$$y(t) = c_1 e^{i\sqrt{\frac{K}{M}}\,t} + c_2 e^{-i\sqrt{\frac{K}{M}}\,t}$$

or equivalently

$$y(t) = c_1\left[\cos\sqrt{\frac{K}{M}}\,t + i\sin\sqrt{\frac{K}{M}}\,t\right]$$
$$+ c_2\left[\cos\sqrt{\frac{K}{M}}\,t - i\sin\sqrt{\frac{K}{M}}\,t\right]$$
$$= (c_1 + c_2)\cos\sqrt{\frac{K}{M}}\,t + i(c_1 - c_2)\sin\sqrt{\frac{K}{M}}\,t$$
$$= A\cos\sqrt{\frac{K}{M}}\,t + B\sin\sqrt{\frac{K}{M}}\,t$$

The function is periodic with period

$$T = 2\pi \Big/ \sqrt{\frac{K}{M}}$$

$$\nu = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{K}{M}} \quad \text{and} \quad \omega = \frac{2\pi}{T} = \sqrt{\frac{K}{M}}$$

The soln is

$$y(t) = A\cos\omega t + B\sin\omega t \Rightarrow \text{Harmonic}$$
$$\equiv D\cos(\omega t - \delta)$$

[Put $A = D\cos\delta$, $B = D\sin\delta$

$$\therefore A\cos\omega t + B\sin\omega t = D\cos\omega t\cos\delta + D\sin\omega t\sin\delta$$
$$= \underbrace{D}_{\text{Amplitude}}\cos(\omega t - \underbrace{\delta}_{\text{phase}})$$

Here $D = \sqrt{A^2 + B^2}$

$$\tan\delta = \frac{B}{A}$$

When damping is present

$$My'' + cy' + ky = 0$$

$$y'' + c/M\, y' + \frac{k}{M} y = 0$$

$$y'' + py' + qy = 0 \qquad \text{where } p = c/M$$
$$q = k/M$$

consider $y = e^{\lambda x}$ as the solⁿ.

Then as before we have the roots

$$\lambda = \left[-p \pm \sqrt{p^2 - 4q}\right]/2$$

$$= -\frac{c}{2M} \pm \frac{\sqrt{c^2/M^2 - 4k/M}}{2}$$

$$= -\frac{c}{2M} \pm \frac{1}{2M}\sqrt{c^2 - 4kM}$$

Let $\Omega = \frac{1}{2M}\sqrt{c^2 - 4kM}$

The general solⁿ is

$$y(t) = c_1 e^{-c/2M\, t + \Omega t} + c_2 e^{-\frac{c}{2M}t - \Omega t}$$

$$= e^{-\frac{c}{2M}t}\left[c_1 e^{\Omega t} + c_2 e^{-\Omega t}\right]$$

As usual we have the three cases

i) Real roots (distinct) $\Omega = +ve$

$\Rightarrow$ overdamped motion

ii) identical roots $\Omega = 0$

Here we have critical damping

(iii) complex roots $\Omega < 0$

$\Rightarrow$ underdamping.

2.6 Nonhomogeneous second order DE with constant coefficients:

$$y'' + P(x)y' + Q(x)y = R(x) \quad \cdots \quad 2.2.1$$

A general sol^n is given by adding a particular sol^n to the gen. sol^n of the corresponding homogeneous eqn.

$$y(x) = y_g(x) + y_p(x) \quad \cdots \quad 2.6.1$$

There is no generally applicable method for finding $y_p(x)$.

But for second order linear DE with constant coefficients and a simple RHS, $y_p(x)$ can often be found by inspection or by assuming a parametrised form as that of $R(x)$. The better method is sometimes called the method of undetermined coefficients. If $R(x)$ contains only polynomial, exponential or sine or cosine terms, then by assuming a trial function for $y_p(x)$ of similar form but which contains a number of undetermined parameters and substituting this trial f^n into eq. 2.2.2 (homogeneous), the parameters can be found and $y_p(x)$ deduced.

standard trial f^nn :-

(i) If $R(x) = ae^{rx}$ then try $y_p(x) = be^{rx}$
if r is not a root of the characteristic eq^n.

If r is a single root
$$y_p(x) = bxe^{rx}$$

If r is a double root
$$y_p(x) = bx^2e^{rx}$$

(ii) If $R(x)$ is a polynomial of degree n

$$R(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

(some a_m may be zero)

try $y_p(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n$

for $\lambda = 0$ ~~→~~ not a root of the characteristic eqn. (CE)

If $\lambda = 0 \rightarrow$ a single root of CE.

then $y_p(x) = x(b_0 + b_1 x + \cdots + b_n x^n)$

If $\lambda = 0 \rightarrow$ double root of CE.

$y_p(x) = x^2(b_0 + b_1 x + \cdots + b_n x^n)$

(iii) If $R(x) = a_1 \sin rx + a_2 \cos rx$ (a_1 or a_2 may be zero)

then try $y_p(x) = b_1 \sin rx + b_2 \cos rx$

if $ir \neq$ root of CE

try $y_p(x) = x b_1 \sin rx + x b_2 \cos rx$

if $ir \rightarrow$ single root of CE

ir can not be double root [$\pm ir$ will appear]

(iv) If $R(x) \Rightarrow$ combination of the above f^n.

try $y_p(x)$ also according as suitable combinations.

To find a particular integral of

$$y'' - 2y' + y = e^{x}$$

From previous discussion a trial guess would be

$$y(x) = be^{x}$$

But the complementary function is

$$y(x) = (c_1 + c_2 x)e^{x}$$

We find that e^{x} and xe^{x} are already in y_c.

Thus multiplying our first guess by the lowest necessary integral power of x such that it does not appear in $y_c(x)$, we try bx^2e^{x}

substitution yields

$$y' = b\left[2x + x^2\right]e^{x}$$

$$y'' = b\left[2x + x^2 + 2 + 2x\right]e^{x}$$

$$\therefore y'' - 2y' + y$$

$$= b\left[\cancel{4x} + \cancel{x^2} + 2 - 2(\cancel{2x} + \cancel{x^2}) + \cancel{x^2}\right]e^{x}$$

$$= 2be^{x}$$

R.H.S $= e^{x}$ $\qquad \therefore b = \frac{1}{2}$

Thus the particular integral is $\frac{1}{2}x^2e^{x}$

2.7 Variation of Parameters

This method is useful for finding particular integrals for linear ODE with variable or constant coefficients. However, this method requires ~~the~~ knowledge of entire complementary functions and not just a part of it.

We have the eqn [second order linear]

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y = R(x) \quad \ldots \quad 2.2.1$$

The general solution of the homogeneous eq^n is

$$y_c(x) = c_1 y_1(x) + c_2 y_2(x)$$

where y_1 and y_2 are known.

We now assume that the particular integral be

$$y_p(x) = k_1(x)\, y_1(x) + k_2(x)\, y_2(x) \quad \ldots \quad 2.7.1$$

i.e. the particular integral be like that of the complementary f^n but the constants replaced by functions. (2.7.1) will not satisfy the eq^n (2.2.1) with $R(x) = 0$ but with suitable choice of $k_1(x)$ & $k_2(x)$ this can be made equal to $R(x)$.

Thus producing a particular integral ~~so~~ and not a complementary function.

Here we have two functions k_1 and k_2 but there is only one eq^n, namely 2.2.1 to satisfy. We have to impose further conditions. The simplest choice is by differentiation of 2.7.1.

$$\frac{dy_p}{dx} = \frac{dk_1}{dx} y_1 + k_1 \frac{dy_1}{dx} + \frac{dk_2}{dx} y_2 + k_2 \frac{dy_2}{dx}$$

And we impose the restriction that

$$\frac{dk_1}{dx} y_1 + \frac{dk_2}{dx} y_2 = 0 \quad \text{.........} \; 2.7.2$$

The following analysis shows that we can make this restriction.

We have (with 2.7.2)

$$\frac{dy_p}{dx} = k_1 \frac{dy_1}{dx} + k_2 \frac{dy_2}{dx}$$

or $$\frac{d^2y_1}{dx^2} = \frac{dk_1}{dx}\frac{dy_1}{dx} + k_1 \frac{d^2y_1}{dx^2} + \frac{dk_2}{dx}\frac{dy_2}{dx} + k_2 \frac{d^2y_2}{dx^2}$$

substitution in (2.2.1) gives

$$k_1 \frac{d^2y_1}{dx^2} + k_2 \frac{d^2y_2}{dx^2} + \frac{dk_1}{dx}\frac{dy_1}{dx} + \frac{dk_2}{dx}\cdot\frac{dy_2}{dx}$$
$$+ P(x)\left\{k_1 \frac{dy_1}{dx} + k_2 \frac{dy_2}{dx}\right\} + Q(x)(y_1 + y_2) = R(x)$$

or $$k_1\left[\frac{d^2y_1}{dx^2} + P(x)\frac{dy_1}{dx} + Q(x)y_1\right] + k_2\left[\frac{d^2y_2}{dx^2} + P(x)\frac{dy_2}{dx} + Q(x)y_2\right]$$
$$+ \frac{dk_1}{dx}\frac{dy_1}{dx} + \frac{dk_2}{dx}\frac{dy_2}{dx} = R(x)$$

since y_1 & y_2 are solutions of homogeneous DE,

[ ] terms = 0

Thus

$$\frac{dk_1}{dx}\frac{dy_1}{dx} + \frac{dk_2}{dx}\frac{dy_2}{dx} = R(x) \quad \ldots\ldots \quad 2.7.3$$

Also we have $\frac{dk_1}{dx} y_1 + \frac{dk_2}{dx} y_2 = 0 \quad \ldots \quad 2.7.2$

We have to find the solution of the simultaneous eq^n 2.7.2 & 2.7.3 to get k_1' and k_2'

From sol^{n}s of linear eq^n we know

$$\frac{dk_1}{dx} = \begin{vmatrix} R(x) & \frac{dy_2}{dx} \\ 0 & y_2 \end{vmatrix} \Bigg/ \begin{vmatrix} \frac{dy_1}{dx} & \frac{dy_2}{dx} \\ y_1 & y_2 \end{vmatrix}$$

$$= \frac{R(x)\, y_2}{y_2 \frac{dy_1}{dx} - y_1 \frac{dy_2}{dx}} = -\frac{y_2 R(x)}{y_1 y_2' - y_1 y_2'} \quad \ldots \quad 2.7.4$$

$$\frac{dk_2}{dx} = \begin{vmatrix} \frac{dy_1}{dx} & R(x) \\ y_1 & 0 \end{vmatrix} \Bigg/ \begin{vmatrix} \frac{dy_1}{dx} & \frac{dy_2}{dx} \\ y_1 & y_2 \end{vmatrix}$$

$$= \frac{-y_1 R(x)}{y_2 \frac{dy_1}{dx} - y_1 \frac{dy_2}{dx}} = \frac{y_1 R(x)}{y_1 y_2' - y_2 y_1'} \quad \ldots \quad 2.7.5$$

The denominator is the Wronskian of y_1 & y_2 [$= W(x)$] and it is non zero as y_1 & y_2 are linearly indep. sol^{n}s. of 2.2.2.

$$\therefore \frac{dk_1}{dx} = -y_2\, R(x)/W(x)$$

$$\frac{dk_2}{dx} = \frac{y_1 R(x)}{W(x)}$$

$$k_1(x) = -\int y_2(x)\, R(x)/W(x)\, dx$$

$$k_2 = +\int y_1(x)\, R(x)/W(x)\, dx.$$

Resulting in the particular soln.

$$y_p(x) = -y_1\int \frac{y_2 R(x)}{W(x)}dx + y_2\int \frac{y_1 R(x)}{W(x)}dx \quad \ldots\ 2.7.6$$

The general solution is given by

$$y_g = y_c + y_p = c_1 y_1(x) + c_2 y_2(x) + k_1 y_1(x) + k_2 y_2(x)$$

If the constants of integration are included in $k_1(x)$ & $k_2(x)$ we simply recover the complementary f^n in addition to the particular integral.

Ex.
$$\frac{d^2y}{dx^2} + y = \operatorname{cosec} x \quad \text{with } y(0) = y(\pi/2) = 0$$

complementary function

$$y_c(x) = c_1 \sin x + c_2 \cos x$$

Assume $y_p(x) = k_1(x)\sin x + k_2(x)\cos x$

constrain $\frac{dk_1}{dx}\sin x + \frac{dk_2}{dx}\cos x = 0$

which give $\frac{dk_1}{dx}\cos x - \frac{dk_2}{dx}\sin x = \operatorname{cosec} x$

solving $\frac{dk_1}{dx} = \cos x \operatorname{cosec} x = \cot x$

$$\frac{dk_2}{dx} = -\sin x \operatorname{cosec} x = -1$$

$\therefore k_1 = \int \cot x\, dx = \ln(\sin x)$

$k_2 = -x$

$\therefore$ Soln of ODE $= [c_1 + \ln(\sin x)]\sin x + [c_2 - x]\cos x$

Boundary conditions yield $c_1 = 0;\ c_2 = 0$

Hence $y(x) = \ln(\sin x)\sin x - x\cos x$

2.8 Singular Points of a DE :-

An ODE of the form

$$y'' = f(x, y, y') \quad \ldots\ldots\ldots \quad 2.8.1$$

can be solved by using various techniques of which some are described before.

consider any point $x = x_0$

If y' and y take on all finite value, at $x = x_0$ and y'' remains finite, the point $x = x_0$ is an ordinary point.

If y'' becomes infinite at $x = x_0$, the point is called "singular".

Consider the homogeneous eq.

$$y'' + P(x)\, y' + Q(x)\, y = 0 \quad \ldots\ldots \quad 2.2.2$$

$x = x_0 \Rightarrow$ ordinary point

if $P(x)$ & $Q(x) \rightarrow$ remain finite

If $P(x)$ or $Q(x)$ or both diverges at $x = x_0$ the point is singular.

We have two kinds of singularity.

(i) If $P(x)$ or $Q(x)$ diverges at $x = x_0$ but $(x - x_0)\,P(x)$ and $(x - x_0)^2 Q(x)$ remain finite at $x = x_0$, then $x = x_0$ is called a regular or nonessential singular point. This is removable.

(ii) If $P(x)$ diverges faster than $\frac{1}{(x-x_0)}$ so that $(x-x_0)P(x)$ goes to infinity as $x \to x_0$ or $Q(x)$ diverges faster than $\frac{1}{(x-x_0)^2}$ so that $(x-x_0)^2 Q(x)$ diverges as $x \to x_0$, then $x = x_0$ is labelled an essential singularity or irregular singularity.

These definitions hold for all finite points x_0. For $x_0 \to \infty$, the analysis should be done according to analysis similar to complex. We set $x = \frac{1}{z}$ in the ODE and then let $z \to 0$.

Here $$\frac{d}{dx} y(x) = \frac{dy(z^{-1})}{dz} \cdot \frac{dz}{dx}$$

$$= -\frac{1}{x^2} \frac{d}{dz}(z^{-1}) = -z^2 \frac{d}{dz}(z^{-1})$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \cdot \frac{dy(x)}{dx} = \frac{d}{dz}\left[-z^2 \frac{d}{dz}(z^{-1})\right] \frac{dz}{dx}$$

$$= -z^2\left[-2z \frac{d}{dz}(z^{-1}) - z^2 \frac{d^2}{dz^2}(z^{-1})\right]$$

$$= 2z^3 \frac{d}{dz}(z^{-1}) + z^4 \frac{d^2y(z^{-1})}{dz^2}$$

Hence the ODE becomes

$$z^4 \frac{d^2y}{dz^2}(z^{-1}) + (2z^3 - z^2)\, P(z^{-1}) + Q(z^{-1})\, y(z^{-1}) = 0$$

or $$\frac{d^2 y(z^{-1})}{dz^2} + \left(\frac{2z-1}{z^2}\right) P(z^{-1}) + \frac{Q(z^{-1})}{z^4}\, y(z^{-1}) = 0$$

The behaviour at $x = \infty$ i.e. $z = 0$ then depends on the behaviour of the coefficients $\frac{(2z^3 - z^2)}{z^4} P(z^{-1})$ i.e. $\frac{(2z-1)}{z^2} P(z^{-1})$ & $\frac{Q(z^{-1})}{z^4}$ as $z \to 0$

If the coefficients remain finite at $z = 0$ then point $x = \infty$ is an ordinary point. If they diverge less rapidly than $\frac{1}{z}$ or $\frac{1}{z^2}$ respectively, then $x = \infty$ is a regular singular point, otherwise it is an essential singular point.

We shall discuss cases occuring in physics.

Ex: Consider Bessel's eq^n $x^2 y'' + x y' + (x^2 - n^2) y = 0$

or $y'' + \frac{1}{x} y' + \left(1 - \frac{n^2}{x^2}\right) y = 0$

Here $P(x) = \frac{1}{x}$; $Q(x) = 1 - \frac{n^2}{x^2}$

$x = 0$ is a removable singular point. There are no other singularities in the finite range. But as $x \to \infty$ or $z = \frac{1}{x} \to 0$ we have the coeffs.

$$\frac{2z - P(z^{-1})}{z^2} \quad \& \quad \frac{Q(z^{-1})}{z^4} \Rightarrow \frac{2z - z}{z^2} \quad \& \quad \frac{1 - n^2 z^2}{z^4}$$

The latter diverges as $\frac{1}{z^4}$ and hence $x = \infty$ or $z = 0$ is an essential singular point of Bessel's eq^n.

We shall give the singular points for some standard second order ODE.

Eqn	Regular singularity	Irregular singularity
1. Hypergeometric $x(x-1)y'' + [(1+a+b)x - c]y' + aby = 0$	$0, 1, \infty$	—
2. Legendre* $(1-x^2)y'' - 2xy' + l(l+1)y = 0$	$-1, 1, \infty$	—
3. Chebyshev. $(1-x^2)y'' - xy' + n^2 y = 0$	$-1, 1, \infty$	—
4. . . . . . . hypergeometric $xy'' + (c-x)y' - ay = 0$	0	∞
5. Bessel $x^2y'' + xy' + (x^2 - n^2)y = 0$	0	∞
6. Laguerre* $xy'' + (1-x)y' + ay = 0$	0	∞
7. Hermite $y'' - 2xy' + 2\alpha y = 0$	—	∞
8. simple harmonic oscillator $y'' + \omega^2 y = 0$	—	∞

* The corresponding associated eqns have the same singular points.

2.9 Series Solution : Frobenius Method :

This method gives one solution of the linear second order homogeneous DE. The method, a series expansion, always works provided the point of expansion is no worse than a regular singular point. In physics this condition is always almost satisfied.

We have the general homogeneous ODE

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y = 0 \quad - - - - - \quad 2.2.2$$

Here we develop the method to obtain at least one solution of 2.2.2. Next we show that a second sol^n exists and prove that no third independent sol^n exists.

Thus the most general sol^n of 2.2.2 is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) \quad \cdots\cdots \quad 2.9.1$$

To illustrate consider the linear oscillator eq^n.

$$\frac{d^2y}{dx^2} + \omega^2 y = 0 \quad \cdots\cdots\cdots \quad 2.9.2$$

The known sol^{n}s are $\sin\omega x$ & $\cos\omega x$

We try

$$y(x) = x^{k}\left(a_0 + a_1 x + a_2 x^2 + \ldots\ldots\right)$$

$$= \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda} \qquad a_0 \neq 0 \quad \ldots 2.9.3$$

exponent k and all coefficients a_λ are to be found. Here k need not be an integer.

Here $$\frac{dy}{dx} = \sum_{\lambda=0}^{\infty} (k+\lambda)\, a_\lambda\, x^{k+\lambda-1}$$

$$\frac{d^2y}{dx^2} = \sum_{\lambda=0}^{\infty} (k+\lambda)(k+\lambda-1)\, a_\lambda\, x^{k+\lambda-2}$$

substituting in 2.9.2

$$\sum_{\lambda=0}^{\infty} (k+\lambda)(k+\lambda-1)\, a_\lambda\, x^{k+\lambda-2} + \omega^2 \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda} = 0$$

coefficients of each power of x will vanish.

Equating the lowest power of x i.e. for $\lambda = 0$ $[x^{k-2}]$ we have

$$k(k-1)\, a_0 = 0$$

$$\because a_0 \neq 0 \qquad k(k-1) = 0$$

The eqn coming from the lowest power of x is called the "indicial equation".

Here indicial equation is

$$k(k-1) = 0$$

$$\therefore k = 0 \text{ or } 1.$$

Also comparing the coefficient of x^{k+j} $j \geq 0$

for $\lambda = j+2$ (say) we have

$$a_{j+2}(k+j+2)(k+j+1) + \omega^2 a_j = 0$$

or $$a_{j+2} = -a_j \frac{\omega^2}{(k+j+2)(k+j+1)} \quad \cdots\cdots \quad 2.9.4$$

This is a two term recurrence relation and [there may be three terms recurrence relation]

we get a_{j+2}, a_{j+4}, ... etc, given a_j.

Thus we get $a_2, a_4, a_6, \ldots$ even terms.

and $a_1, a_3, \ldots$ i.e. the odd terms are ignored in 2.9.4. Since a_1 is arbitrary, put $a_1 = 0$, this gives $a_3, a_5, \ldots = 0$

This gives us a solution. The powers of x ignored in first soln will appear in the second solution.

Let us consider the solution for $k=0$

The recurrence relation gives

$$a_{j+2} = -a_j \frac{\omega^2}{(j+2)(j+1)} \quad \cdots\cdots \quad 2.9.5$$

which leads to

$$a_2 = -a_0 \frac{\omega^2}{1.2} = -\frac{\omega^2}{2!} a_0$$

$$a_4 = -a_2 \frac{\omega^2}{3.4} = +\frac{\omega^4}{4!} a_0$$

$$a_6 = -a_4 \frac{\omega^2}{5.6} = -\frac{\omega^6}{6!} a_0 \quad \text{and so on.}$$

Thus by inspection

$$a_{2n} = (-1)^n \frac{\omega^{2n}}{(2n)!} a_0 \quad \dots\dots\dots\dots \quad 9.2.6$$

$$\therefore \underset{k=0}{y(x)} = a_0 \left[1 - \frac{(\omega x)^2}{2!} + \frac{(\omega x)^4}{4!} + \dots\dots \right]$$

$$= a_0 \cos \omega x$$

For $k = 1$

$$a_{j+2} = -a_j \frac{\omega^2}{(j+3)(j+2)} \quad \dots\dots\dots \quad 9.2.7$$

$$\therefore a_2 = -a_0 \frac{\omega^2}{2.3} = -\frac{\omega^2}{3!} a_0$$

$$a_4 = -a_2 \frac{\omega^2}{4.5} = \frac{\omega^4}{5!} a_0$$

$$a_6 = -\frac{\omega^6}{7!} a_0$$

By inspection

$$a_{2n} = (-1)^n \frac{\omega^{2n}}{(2n+1)!} a_0$$

Thus for this choice $k=1$

$$y(x) = a_0 x \left[1 - \frac{(\omega x)^2}{3!} + \frac{(\omega x)^4}{5!} - \frac{(\omega x)^6}{7!} + \dots\dots \right]$$

$$= \frac{a_0}{\omega} \left[(\omega x) - \frac{(\omega x)^3}{3!} + \frac{(\omega x)^5}{5!} - \frac{(\omega x)^7}{7!} + \dots \right]$$

$$= \frac{a_0}{\omega} \sin \omega x$$

The method of substitution by series is called the Frobenius method. One should note

i) The series solution should be substituted back in the DE to see that it works.

ii) The series must converge over the region of interest [In Legendre DE the series ~~dno~~ does not converge.]

Expansion about $x = x_0$

Instead of expanding about $x = 0$. one can make an expansion about $x = x_0$, then

$$y(x) = \sum_{\lambda=0}^{\infty} a_\lambda (x - x_0)^{k+\lambda} \quad \cdots \cdots \quad a_0 \neq 0$$

For ~~Le~~ Legendre, chebyshev and hypergeometric eqn the choice $x_0 = 1$ has some advantage. x_0 should not be an essential singularity where Frobenius method possibly fails. For x_0 as an ordinary point or a regular singular point, the series solution will be valid where it converges.

2.10. Limitation Of Series Approach: Bessel's eq^n:

consider

$$x^2y''+xy'+(x^2-n^2)y=0 \quad \ldots \quad 2.10.1$$

Assume $\quad y=\sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda}$

$$\therefore y'=\sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda) x^{k+\lambda-1}$$

$$y''=\sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)(k+\lambda-1)x^{k+\lambda-2}$$

From 2.10.1

$$\sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)(k+\lambda-1)x^{k+\lambda}+\sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)x^{k+\lambda}+\sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda+2}$$

$$-\sum_{\lambda=0}^{\infty} a_\lambda n^2 x^{k+\lambda}=0 \quad \ldots \quad 2.10.2$$

Equating the lowest power of x appearing in LHS
i.e. $\lambda=0$

$$a_0\left[k(k-1)+k-n^2\right]=0 \quad \ldots \quad 2.10.3$$

$\because a_0\neq 0 \quad$ so $k^2-n^2=0$

$$\text{or } k=\pm n \quad \ldots \quad 2.10.4$$

Consider the next power of x i.e. x^{k+1} and equating coefficients

$$a_1\left[(k+1)(k)+(k+1)-n^2\right]=0$$

or $a_1\left[(k+1)^2-n^2\right]=0$

or $a_1\left[(k+1-n)(k+1+n)\right]=0$

For $k=\pm n$ neither $k+1-n$ nor $k+1+n$ vanishes and we must require $a_1=0$

[Note $k=\pm n=-\frac{1}{2}$ are exceptions]

Now equate the coefficients of x^{k+j}. We set $\lambda = j$ for 1st, 2nd and fourth set of 2.10.2 and $\lambda = j-2$ in the third term, thus

$$a_j\left[(k+j)(k+j-1) + (k+j) - n^2\right] + a_{j-2} = 0$$

For $k = n$

$$a_j\left[(n+j)(n+j-1) + (n+j) - n^2\right] + a_{j-2} = 0$$

or $a_j\left[(n+j)^2 - n^2\right] + a_{j-2} = 0$

or $a_j\left[j(j+2n)\right] + a_{j-2} = 0$

changing j to $j+2$

$$a_{j+2} = \frac{-a_j}{(j+2)(2n+j+2)} \quad \ldots\ldots\ldots \quad 2.10.5$$

This is the two term recurrence relation.

Thus $a_2 = -a_0 \dfrac{1}{2.(2n+2)} = -a_0 \dfrac{1}{2^2(n+1)}$

$$= -a_0 \frac{n!}{2^2\, 1!(n+1)!}$$

$$a_4 = -a_2 \frac{1}{4(2n+4)} = -a_2 \frac{1}{4.2(n+2)}$$

$$= +a_0 \frac{1}{2^2.2^2 1!.2(n+2)} \frac{n!}{(n+1)!} = a_0 \frac{n!}{2^4\, 2!(n+2)!}$$

$$a_6 = -a_4 \frac{1}{6(2n+6)} = -a_0 \frac{n!}{2^4.2!(n+2)!} \cdot \frac{1}{2^2.3(n+3)}$$

$$= -a_0 \frac{n!}{2^6\, 3!(n+3)!} \quad \text{and so on}$$

and in general

$$a_{2p} = (-1)^p \frac{a_0\, n!}{2^{2p}.p!(n+p)!} \quad \ldots\ldots \quad 2.10.6$$

Inserting these coefficients in our assumed series solution we have

$$y(x) = a_0 x^n \left[1 - \frac{n!\,x^2}{2^2 \cdot 1!\,(n+1)!} + \frac{n!\,x^4}{2^4 \cdot 2!\,(n+2)!} - \cdots \right.$$
$$\left. + (-1)^p \frac{n!\,x^{2p}}{2^{2p} \cdot p!\,(n+p)!} + \cdots \right] \quad \ldots\ldots 2.10.7$$

or $$y(x) = a_0 \sum_{j=0}^{\infty} (-1)^j \frac{n!\,x^{n+2j}}{2^{2j} \cdot j!\,(n+j)!}$$

$$= a_0 2^n n! \sum_{j=0}^{\infty} (-1)^j \left(\frac{x}{2}\right)^{n+2j} \frac{1}{j!\,(n+j)!}$$

$$= a_0 2^n n! \sum_{j=0}^{\infty} (-1)^j \frac{1}{j!\,(n+j)!} \left(\frac{x}{2}\right)^{n+2j} \quad \ldots\ldots 2.10.8$$

$$= a_0 . 2^n . n!\; J_n(x) \quad \ldots\ldots\ldots 2.10.9$$

$J_n(x)$ is defined as Bessel f^n of first kind of order n (integral). $J_n(x)$ has either even or odd symmetry.

When $k=-n$ and n is not an integer, we may generate a second distinct series to ~~the~~ be labeled as $J_{-n}(x)$. However, when $-n$ is a negative integer, we get into trouble.

The recurrence relation for a_j's is given by eq. 2.10.5 but with $2n$ replaced by $-2n$. Then when $j+2 = 2n$ or $j = 2(n-1)$, the coefficient a_{j+2} blows up and we have no series solution. This can be remedied by considering (2.10.8)

$$J_{-n}(x) = \sum_{j=0}^{\infty} (-1)^j \frac{1}{j!(-n+j)!} \left(\frac{x}{2}\right)^{-n+2j}$$

since n is integer $(j-n)! = \infty$ for $j = 0, 1, \cdots (n-1)$.

Thus the series should start from $j=n$.

Replacing j by $j+n$ we get

$$J_{-n}(x) = \sum_{j=0}^{\infty} (-1)^{j+n} \frac{1}{(j+n)!} \frac{1}{j!} \left(\frac{x}{2}\right)^{n+2j}$$

Thus $J_n(x) = (-1)^n J_n(x)$ (integral n) ...2.10.10

Thus we get from series substitution two solutions for harmonic oscillator and for Bessel's eq^n (two solutions if $n \neq$ integer) and we can not always get solution by series substitution.

symmetry of solutions!

$y(x)$ is even if $y(x) = y(-x)$

$y(x)$ is odd if $y(x) = -y(-x)$

This is a consequence of the DE.

Let $\mathcal{L}(x)\, y(x) = 0$

$\mathcal{L}(x) \Rightarrow$ differential operator

For linear oscillator

$\mathcal{L}(x) = \mathcal{L}(-x) \Rightarrow$ even parity op.

If $\mathcal{L}(x) = -\mathcal{L}(x) \Rightarrow$ odd parity op.

$\mathcal{L}(x)\, y(x) = 0$

$x \rightarrow -x$

$\mathcal{L}(-x)\, y(-x) = 0$

$\pm \mathcal{L}(x)\, y(-x) = 0$

$+ \rightarrow$ if $\mathcal{L}(x)$ is even

$- \rightarrow$ if $\mathcal{L}(x)$ is odd.

Clearly if $y(x)$ is a solution of DE then $y(-x)$ is also a sol^n. And any sol^n may be written as
$$y(x) = \underbrace{\frac{1}{2}\{y(x)+y(-x)\}}_{\text{even } sol^n} + \underbrace{\frac{1}{2}\{y(x)-y(-x)\}}_{\text{odd } sol^n}$$

We see from the eq^ns that Legendre, Bessel, Chebyshev, SH oscillator and Hermite eq^ns all exhibit even parity. Solutions of all of these may be represented by series with even powers of x or separate series of odd powers of x. The Laguerre differential operator has neither even nor odd symmetry, hence its sol^ns cannot yield even or odd parity.

Regular and Irregular Singularity

The success of the series substitution method depends on the roots of the indicial eq^n and the degree of singularity of the coefficients in the DE.

Consider the following ensemble:

$$y'' - \frac{6}{x^2}y = 0 \quad \ldots \quad 2.10.11$$

$$y'' - \frac{6}{x^3}y = 0 \quad \ldots \quad 2.10.12$$

$$y'' + \frac{1}{x}y' - \frac{a^2}{x^2}y = 0 \quad \ldots \quad 2.10.13$$

$$y'' + \frac{1}{x^2}y' - \frac{a^2}{x^2}y = 0 \quad \ldots \quad 2.10.14$$

For 2.10.11 the series substitution
$$y = \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda}$$ yields

$$\sum_{\lambda=0}^{\infty}(k+\lambda)(k+\lambda-1)a_\lambda x^{k+\lambda-2} - 6\sum_{\lambda} a_\lambda x^{k+\lambda-2} = 0$$

Equating lowest power of x (i.e. $\lambda=0$ term) we get

$$k(k-1) - 6 = 0$$

$$\text{or } k^2 - k - 6 = 0$$

is the indicial equation giving

$$(k-3)(k+2) = 0$$

$$\text{or } k = 3, -2$$

This eq^n is homogeneous in x [as $\frac{d^2}{dx^2}$ behave as x^{-2}] there is no recurrence relation. and $a_\lambda = 0$ for $\lambda > 0$

But we have two solutions x^{-2} and x^{+3}.

Eq^n 2.10.12 has indicial eqn $-6a_0 = 0$ with no solution at all as $a_0 \neq 0$.

The series sol^n worked for 2.10.11 which has a regular singularity at origin but breaks down for 2.10.12 which has an irregular singularity at the origin.

For 2.10.13 and additional term y'/x is present.

The Indicial eqn is

$$k(k-1) + k - a^2 = 0$$

$$\text{i.e. } k^2 - a^2 = 0$$

$$\text{or } k = \pm a$$

There is no recurrence relation but x^a and x^{-a} are two solutions.

considering eqn 2.10.14, the indicial eqn is (only y' term contributing)

$$k = 0$$

The eqn is after power series substitution

$$\sum a_\lambda (k+\lambda)(k+\lambda-1)x^{k+\lambda-2} + \sum_\lambda a_\lambda (k+\lambda) x^{k+\lambda-3}$$

$$- \sum_\lambda a_\lambda a^2 x^{k+\lambda-2} = 0$$

$$a_j (k+j)(k+j-1) + a_{j+1}(k+j+1) - a_j a^2 = 0$$

or $a_j \left[(k+j)(k+j-1) - a^2 \right] + a_{j+1}(k+j+1) = 0$

Setting $k = 0$

$$a_j \left(j(j-1) - a^2 \right) + a_{j+1}(j+1) = 0$$

or $a_{j+1} = a_j \dfrac{a^2 - j(j-1)}{j+1}$ 2.10.15

unless a is selected in such a way that the series terminates.

we have,

$$\lim_{j\to\infty} \left| \frac{a_{j+1}}{a_j} \right| = \lim_{j\to\infty} \frac{a^2 - j(j-1)}{(j+1)} = \lim_{j\to\infty} \left[\frac{a^2}{j+1} + \frac{j(j-1)}{j+1} \right.$$

$$\xrightarrow[j\to\infty]{} j^2/j = \infty$$

Hence the series soln diverges for all $x \neq 0$.

Thus the method worked for a regular singularity in DE but failed when the singularity is irregular as in eqn 2.10.14.

Fuchs's Theorem:

This gives the answer whether series substitution yields solⁿ or not.

At least one power series solution exists provided the expansion is taken about an ordinary point or at a regular singular point.

The series substitution fails if the expansion takes place about an irregular or essential singularity.

Note: Important DE of math. physics have no irregular or essential singularity.

∞ seems to be a singularity for all the eqⁿs. Legendre eqⁿ which as a regular singularity at ∞ has a convergent series soln. for -ve powers of argument. But Bessel's eqⁿ with an irregular singularity at ∞ yields asymptotic series. Although useful, these asymptotic solutions are technically divergent.

Summary:

If we are expanding about an ordinary point or at worst about a regular singularity, the series substitution approach will yield at least one solution (Fuchs Th).

Whether we get one or two distinct solutions depends on the roots of the indicial eqⁿ.

1) If the roots of the indicial eqⁿ are equal, we can obtain only one solution by this series substitution method.

2) If the two roots differ by a nonintegral number, two independent solutions may be obtained.

3) If the two roots differ by an integer, the larger of the two will yield a solution. The smaller may or may not give a sol^n, depending on the behaviour of the coefficients. For linear oscillator we get two solutions; for Bessel's eq^n; only one solution.

2.11. Legendre Eq^n:

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \quad \ldots\ldots \quad 2.11.1$$

Divide by $(1-x^2)$ we get

$$y'' - \frac{2x}{1-x^2}y' + \frac{n(n+1)}{1-x^2}y = 0$$

comparison with standard eq^n gives

$$P(x) = -\frac{2x}{1-x^2} \; ; \quad Q(x) = \frac{n(n+1)}{1-x^2}$$

$$P(x) = \frac{-2x}{(1-x)(1+x)} \; ; \quad Q(x) = \frac{n(n+1)}{(1-x)(1+x)}$$

Changing over to complex numbers z we find that $P(z)$ & $Q(z)$ are analytic at $z=0$, which is an ordinary point; having singularities at $z = \pm 1$

At $z=1$ both $(z-1)P(z)$ and $(z-1)^2 Q(z)$ are analytic, hence $z=1$ is a regular singular point similarly $z=-1$ is another regular singular point.

In a similar way putting $\omega = \frac{1}{z}$ and letting $\omega = 0$ we can show that $\omega = 0$ is a regular singular point i.e. $z = \infty$.

put $y = \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda}$

then

$$\sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)(k+\lambda-1)\{x^{k+\lambda-2} - x^{k+\lambda}\}$$

$$-2\sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda) x^{k+\lambda} + n(n+1)\sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda} = 0$$

or $$\sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)(k+\lambda-1) x^{k+\lambda-2}$$

$$- \sum_{\lambda=0}^{\infty} a_\lambda \{(k+\lambda)(k+\lambda-1) + 2(k+\lambda) - n(n+1)\} x^{k+\lambda} = 0$$

.... (2.11.2)

Equate the lowest power of x on both sides

$\Rightarrow k(k-1) = 0 \Rightarrow$ Indicial eq^n

$\therefore k = 0, 1$

From (2.11.2) equate the power of x^{k+j} on both sides.

$\therefore a_{j+2}(k+j+2)(k+j+1) - a_j\{(k+j)(k+j-1) + 2(k+j) - n(n+1)\} = 0$

For $k=0$

$a_{j+2}\{(j+1)(j+2)\} = a_j\{j(j+1) - n(n+1)\}$

or $$a_{j+2} = a_j \frac{j(j+1) - n(n+1)}{(j+1)(j+2)} \quad \ldots\ldots (2.11.3)$$

And the series contain only even powers of x

$$\therefore y_{even} = a_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n-2)(n+1)(n+3)}{4!}x^4 - \ldots\right] \quad \ldots 2.11.4$$

Odd powers of x do not appear here and we can put $a_1 = 0 \Rightarrow a_3, a_5, \ldots$

$\left[a_2 = a_0\left\{-\frac{n(n+1)}{2!}\right\};\ a_4 = a_2 \frac{2.3 - n(n+1)}{3.4} = a_0 . \frac{n(n+1)}{2!}\left\{\frac{n(n+1)-6}{3.4}\right\}\right.$

$\left. = a_0 \frac{n(n+1)(n+3)(n-2)}{4!} \text{ and so on}\right]$

For $k=1$

$$a_{j+2}(j+3)(j+2) = a_j\{(j+1)j + 2(j+1) - n(n+1)\}$$

or $$a_{j+2} = a_j \frac{(j+1)(j+2) - n(n+1)}{(j+2)(j+3)}$$

one can write

$$\therefore y_{odd} = a_0\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!}x^5 \cdots\cdots\right] \quad \text{--- } 2.11.5$$

$[a_1 = 0]$

$$\left[a_2 = a_0 \frac{2 - n(n+1)}{2\cdot 3} = -a_0\frac{(n-1)(n+2)}{3!}\right.$$

$$a_4 = a_2 \frac{3\cdot 4 - n(n+1)}{4\cdot 5} = a_0\frac{(n-1)(n+2)}{5!}\{n(n+1) - 12\}$$

$$\left. = a_0 \frac{(n-1)(n-3)(n+2)(n+4)}{5!} \quad \text{and so on}\right]$$

From the property of ∞ series one can show that y_{even} / y_{odd} diverges at $x = \pm 1$ [This can be proved from ratio test]

The two solutions y_{even} and y_{odd} are linearly independent, the ratio being not constant. The general solution is $y(x) = y_{even}(x) + y_{odd}(x)$

<u>Imp</u> General solution for integer n

Here $n = 0, 1, 2, \cdots\cdots$ as appearing in QM for hydrogen problem in quantisation of angular momentum.

In this case the recurrence relation

$$a_{j+2} = a_j \frac{j(j+1) - n(n+1)}{(j+1)(j+2)}$$

gives $$a_{n+2} = a_n \frac{n(n+1) - n(n+1)}{(n+1)(n+2)} = 0$$

so that the series terminates and we get a polynomial soln. These are called Legendre polynomials $P_n(x)$. The same can be shown for the other solutions. i.e. by a proper choice of n any one of the series can be made terminating.

We can suitably normalise $P_n(x)$ in such a way that $P_n(1) = 1$ and as a consequence $P_n(-1) = (-1)^n P_n(1) = (-1)^n$

we can write

$$P_n(x) = \sum_{\lambda=0}^{N} (-1)^{\lambda} \frac{(2n-2\lambda)!}{2^n \lambda! (n-\lambda)! (n-2\lambda)!} x^{n-2\lambda} \quad \ldots \; 2.11.6$$

$N = \frac{n}{2}$ for even n;

$N = (n-1)/2$ for odd n

some typical Legendre polynomial are

$P_0(x) = 1$; $P_1(x) = x$

$P_2(x) = \frac{1}{2}[3x^2 - 1]$; $P_3(x) = \frac{1}{2}[5x^3 - 3x]$

$P_4(x) = \frac{1}{8}[35x^4 - 30x^2 + 3]$ and so on.

Without proof we state some important formulae for Legendre polynomials.

1. Put $x = \cos\theta$

Then Legendre eqn 2.11.1 can be transformed into

$$\frac{1}{\sin\theta} \frac{d}{d\theta}\left(\sin\theta \frac{dy}{d\theta}\right) + n(n+1)y = 0$$

This is Legendre eqn in spherical polar co-ordinates and it appears in theory of angular momentum in QM.

2. $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n$

This is Rodrigue formula for Legendre Polynomial for $n = +ve$ integer.

3. $P_n(-x) = (-1)^n P_n(x)$

4. $\frac{d P_n(-x)}{dx} = (-1)^{n+1} \frac{d P_n(x)}{dx}$

5. $\int_{-1}^{+1} P_n(x) P_{n'}(x)\, dx = \frac{2}{(2n+1)} \delta_{nn'}$

i.e. $\int_{-1}^{1} P_n^2(x)\, dx = \frac{2}{2n+1}$; $\int_{-1}^{1} P_n(x) P_{n'}(x) dx = 0 \quad n \neq n'$

$\int_{-1}^{1} P_n(x)\, dx = 0$ for $n \neq 0$

2.12. Hermite DE :

The DE

$$y'' - 2xy' + 2\alpha y = 0 \quad \ldots\ldots\ldots 2.12.1$$

appears in the theory of harmonic oscillator, oscillator and also in statistics.

To get a series solⁿ substitute

$$y = \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda}$$

$$\therefore y' = \sum_{\lambda} a_\lambda (k+\lambda) x^{k+\lambda-1}$$

$$y'' = \sum_{\lambda} a_\lambda (k+\lambda)(k+\lambda-1) x^{k+\lambda-2}$$

Thus from 2.12.1 we have

$$\sum_{\lambda} a_\lambda (k+\lambda)(k+\lambda-1) x^{k+\lambda-2} - 2\sum_{\lambda} a_\lambda (k+\lambda) x^{k+\lambda}$$
$$+ 2\alpha \sum_{\lambda} a_\lambda x^{k+\lambda} = 0 \quad \ldots\ldots 2.12.2$$

Equating lowest power of x

$(k+\lambda)(k+\lambda-1) = 0$ with $\lambda = 0$

i.e. $k(k-1) = 0 \Rightarrow$ indicial eq^n 2.12.3

$\therefore k = 0$ or 1.

For $k = 0$

$$a_{j+2}(j+2)(j+1) = a_j[2j - 2\alpha]$$

$$= a_j 2(j-\alpha)$$

or $a_{j+2} = a_j \dfrac{2(j-\alpha)}{(j+1)(j+2)}$ $\quad j = $ even.

Thus $a_2 = \dfrac{a_0(-2\alpha)}{2!} = a_0 \dfrac{2(-\alpha)}{2!}$

$a_4 = a_2 . \dfrac{2(2-\alpha)}{3.4} = a_0 \dfrac{2(-\alpha)2(2-\alpha)}{2!\,3.4} = a_0 \dfrac{2^2(-\alpha)(2-\alpha)}{4!}$

Thus $y_{even} = a_0\left[1 + \dfrac{2(-\alpha)}{2!}x^2 + 2^2\dfrac{(-\alpha)(2-\alpha)}{4!}x^4 + \dots\right]$

. . . . 2.12.4

$a_1 = 0 \; [= a_3, a_5, \dots]$

For $k = 1$ we get the odd terms

$$a_{j+2}(j+3)(j+2) = a_j[2(j+1) - 2\alpha]$$

or $a_{j+2} = a_j \dfrac{2(j+1-\alpha)}{(j+2)(j+3)}$ 2.12.5

$j = $ even

Thus $a_2 = a_0 \dfrac{2(1-\alpha)}{2.3} = a_0 \dfrac{2(1-\alpha)}{3!}$

$a_4 = a_2 \dfrac{2(3-\alpha)}{4.5} = a_0 \dfrac{2^2(1-\alpha)(3-\alpha)}{5!}$

$\vdots$

Thus

$$y_{odd}(x) = a_0\left[x + \frac{2(1-\alpha)}{3!}x^3 + \frac{2^2(1-\alpha)(3-\alpha)}{5!}x^5 + \cdots\right] \quad \ldots 2.12.6$$

$$a_1 = 0 \,[= a_3 = a_5 \ldots \text{etc}]$$

The series solutions are convergent for all x, the ratio of the successive coefficients behaving for large index, like the corresponding ratio in the expansion of $\exp(2x^2)$.

The two series soln may be written in compact form as

$$y_{even}(x) = 1 + \sum_{\lambda=1}^{\infty} \frac{2^\lambda (-\alpha)(2-\alpha)\cdots(2\lambda-2-\alpha)}{(2\lambda)!} x^{2\lambda} \quad \ldots 2.12.7$$

$$y_{odd}(x) = x + \sum_{\lambda=1}^{\infty} \frac{2^\lambda (1-\alpha)(3-\alpha)\cdots\cdots(2\lambda-1-\alpha)}{(2\lambda+1)!} x^{2\lambda+1} \quad \ldots 2.12.8$$

we notice that we get polynomial solutions for α, a nonnegative integer.

For $\alpha = 0$ $\quad y_{even}(x) = 1$

$= 2$ $\quad y_{even}(x) = 1 - 2x^2$

$= 4$ $\quad y_{even}(x) = 1 - 4x^2 + \frac{4}{3}x^4$

for $\alpha = $ odd

$\alpha = 1$. $\quad y_{odd}(x) = x$

$\alpha = 3$ $\quad y_{odd}(x) = x - \frac{2}{3}x^3$

$\alpha = 5$ $\quad y_{odd}(x) = x - \frac{4}{3}x^3 + \frac{4}{15}x^5$

certain multiples of these polynomials are called Hermite polynomials $H_n(x)$

$H_0(x) = 1$

$H_1(x) = x$

$H_2(x) = -2 + 4x^2$

$H_3(x) = -12x + 8x^3$ and so on

In general

$$H_n(x) = n! \sum_{\lambda=0}^{N} \frac{(-1)^\lambda (2x)^{n-2\lambda}}{\lambda!\,(n-2\lambda)!} \quad \ldots\ldots 2.12.9$$

where $N = n/2$ for $n = \text{even}$
$= (n-1)/2$ for $n = \text{odd}$

These polynomials occurs in QM in the harmonic oscillator problem.

Some useful relations for $H_n(x)$

$$H_n(-x) = (-1)^n H_n(x)$$

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right)$$

and

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x)\, dx = 0 \qquad m \neq n$$

$$\int_{-\infty}^{\infty} e^{-x^2} H_n^2(x)\, dx = 2^n n! \sqrt{\pi}$$

To obtain $H_n(x)$ from the soln of the DE we have, one has to transform the DE into what is known as Sturm-Liouville form by multiplying by the integrating factor e^{-x^2} which yields the DE.

$$e^{-x^2} y'' - 2x e^{-x^2} y' + 2\alpha e^{-x^2} y = 0$$

$$\text{or } \left(e^{-x^2} y'\right)' + 2\alpha e^{-x^2} y = 0 \qquad \ldots \quad 2.12.10$$

The solutions of 2.12.10 yields $H_n(x)$

2.13. Second Solutions:

We have seen that solutions of second order homogeneous DE can be obtained using power series substitution. By Fuchs theorem, this is possible provided the power series is an expansion about an ordinary point or a nonessential singularity. There is no guarantee that this approach will yield the two independent solutions we expect from a linear 2nd order differential eqn. Indeed, this technique gave only one solution for Bessel's eqn (n = integer). We now indicate some methods to get the second solution.

The DE is

$$y'' + P(x)y' + Q(x)y = 0 \quad \ldots \quad 2.2.2$$

Let $y_1(x)$ and $y_2(x)$ be the two independent solution then the Wronskian is

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1' \quad \ldots \quad 2.13.1$$

$$\therefore W'(x) = \cancel{y_1' y_2'} + y_1 y_2'' - \cancel{y_2' y_1'} - y_2 y_1''$$

$$= y_1 y_2'' - y_2 y_1''$$

$$= y_1[-P(x) y_2' - Q(x) y_2] - y_2[-P(x) y_1' - Q(x) y_1]$$

$$= -P(x)[y_1 y_2' - y_2 y_1'] = -P(x) W(x)$$

Thus $\quad W' = -P(x) W \quad \ldots \quad 2.13.2$

If $P(x) = 0$ i.e. $\quad \ldots \quad 2.13.3$

$$y'' + Q(x) y = 0$$

The Wronskian $\quad W' = 0$ i.e.

$$W = \text{constant}$$

Since our original DE is homogeneous we may multiply the solutions by arbitrary constants and make the Wronskian $= 1$ (or -1) and $P(x) = 0$ will be satisfied. Finally every linear second order DE can be transformed into the form of eq. 2.13.3 [where $P(x) = 0$].

Let us now assume that we have one solution of eq. 2.2.2. by series substitution or by guess. We now proceed to determine the second solⁿ which is linearly independent to the first one

Eq 2.13.2

$$W' = -P(x) W$$

$$\therefore \frac{W'}{W} = -P(x)$$

$$\therefore \int \frac{W' dx}{W} = \int -P(x) dx$$

or $\ln W(x) = -\int P(x)dx + \text{constant}$

$$\therefore \; W(x) = W(a)\, e^{-\int_a^x P(x)dx} \qquad \ldots\ldots\ldots 2.13.4$$

Note $\Rightarrow$ $W(x) \neq 0$ ~~under~~ unless $W(a) = 0$

i.e. $W(x)$ is either identically zero or never zero.

But $W(x) = y_1 y_2' - y_1' y_2$

$$= y_1^2 \frac{d}{dx}\left(\frac{y_2}{y_1}\right)$$

$$\therefore \; \frac{d}{dx}\left(\frac{y_2}{y_1}\right) = W(a)\frac{\exp\left[-\int_a^x P(x)dx\right]}{y_1^2} \qquad \ldots\ldots 2.13.5$$

$$= W(a)\frac{\exp\left[-\int_a^x P(x_1)dx_1\right]}{y_1^2}$$

Integrating from $x_2 = b$ to $x_2 = x$ we have

$$y_2(x) = y_1(x)\, W(a) \int_b^x \frac{\exp\left[-\int_a^{x} P(x_1)dx_1\right]}{y_1^2(x_2)}\, dx_2$$

$$+ y_1(x)\, y_2(b)/y_1(b) \qquad \ldots\ldots 2.13.6$$

Here a, b are arbitrary constants.

The term $y_1(x)\dfrac{y_2(b)}{y_1(b)}$ may be dropped because it gives nothing new.

Since $W(a)$ is a constant and our sol^n for the homogeneous DE always contain an unknown normalising factor, we set $W(a) = 1$. and write

$$y_2(x) = y_1(x) \int^x dx_2 \frac{\exp\left[-\int^x P(x_1)dx_1\right]}{y_1^2(x_2)} \qquad \ldots 2.13.7$$

The lower limits have been omitted as they yield a constant times the soln. $y_1(x)$ and hence nothing new.

Special case

$P(x) = 0$ then 2.13.7 become

$$y_2(x) = y_1(x) \int^{x} \frac{dx_2}{y_1^2(x_2)} \quad \ldots \ldots \quad 2.13.8$$

Thus by using either 2.13.7 or 2.13.8 we can generate the second independent sol^n of the DE provided the other solution is known.

Ex. 2nd soln of linear oscillator

let $\frac{d^2y}{dx^2} + y = 0$

Here $P(x) = 0$

Let $y_1(x) = \sin x$

then $y_2(x) = \sin x \int^{x} \frac{dx_2}{\sin^2(x_2)}$

$= \sin x \int^{x} \mathrm{cosec}^2(x_2)\,dx_2$

$= \sin x\,[-\cot x] = -\cos x$

which is a second linearly independent sol^n of DE

2.14 Series form of Second Sol^n :-

$$y'' + P(x)y' + Q(x)y = 0 \quad \ldots \ldots \quad 2.2.2$$

let
$$\left.\begin{aligned} P(x) &= \sum_{i=-1}^{\infty} p_i x^i \\ Q(x) &= \sum_{j=-2}^{\infty} q_j x^j \end{aligned}\right\} \quad \ldots \ldots \quad 2.14.1$$

1) The lower limits of summation are selected to create strongest possible regular singularity at the origin. These ~~contain~~ conditions just satisfy Fuchs theorem and leads to a better understanding of it.

2) Develope the first few terms of the power series sol^n.

3) Using this sol^n as y_1 obtain the 2nd y_2 using 2.13.7 & 8.

using 2.14.1 in 2.2.2 we have

$$y'' + (p_{-1}x^{-1} + p_0 + p_1 x + \dots)y' + (q_{-2}x^{-2} + q_{-1}x^{-1} + q_0 + \dots)y = 0$$

in which $x = x_0$ is at worst a regular singular point. If $p_{-1} = q_{-1} = q_{-2} = 0$, it reduces to an ordinary point.

Put $y = \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda}$

substituting we obtain

$$\sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)(k+\lambda-1)x^{k+\lambda-2} + \sum_{i=-1}^{\infty} p_i x^i \sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda) x^{k+\lambda-1}$$

$$+ \sum_{j=-2}^{\infty} q_j x^j \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda} = 0 \quad \dots \dots 2.14.2$$

Assuming $p_{-1} \neq 0$ and $q_{-2} \neq 0$ our indicial eqn is given by $k(k-1) + p_{-1}k + q_{-2} = 0$

which sets the total coefficient of $x^{k-2} = 0$ This gives

$$k^2 + k(p_{-1} - 1) + q_{-2} = 0 \quad \dots \dots \dots \dots 2.14.3$$

The soln s are assumed as

$k = \alpha$

and $k = \alpha - n$ where n is a non negative integers.

If n is not an integer we get two independent series soln.

$$\therefore (k-\alpha)(k-\alpha+n) = 0 \quad \dots \dots \dots 2.14.4$$

or $k^2 + (n - 2\alpha)k + \alpha(\alpha - n) = 0$

$$\therefore \left.\begin{aligned} n - 2\alpha &= p_{-1} - 1 \\ \alpha(\alpha - n) &= q_{-2} \end{aligned}\right\} \quad \dots \dots \dots 2.14.5$$

The known series soln corresponding to larger root $k = \alpha$ is given by

$$y_1(x) = x^\alpha \sum_{\lambda=0}^{\infty} a_\lambda x^\lambda$$

substituting this series solution in 2.13.7 we have

$$y_2(x) = y_1(x) \int^x \frac{\exp\left[-\int_a^{x_2} \sum_{i=-1}^{\infty} p_i x_1^i dx_1\right]}{x_2^{2\alpha}\left[\sum_{\lambda=0}^{\infty} a_\lambda x_2^\lambda\right]^2} dx_2 \quad \ldots 2.14.6$$

where the solutions y_1 and y_2 have been normalised in a way to make $W(a) = 1$

Taking the exponential first

$$\left[\int_a^{x_2} \sum_{i=-1}^{\infty} p_i x_1^i dx_1\right] = p_{-1} \ln x_2 + \sum_{k=0}^{\infty} \frac{p_k}{(k+1)} x_2^{k+1} + \underbrace{f(a)}_{\text{const.}} \quad \ldots 2.14.7$$

Hence

$$\exp\left[-\int_a^{x_2} \sum_{i=-1}^{\infty} p_i x_1^i dx_1\right]$$

$$= \exp\left[-\left[p_{-1} \ln x_2 + \sum_{k=0}^{\infty} \frac{p_k}{k+1} x_2^{k+1} + f(a)\right]\right]$$

$$= \exp\left[-f(a)\right] \cdot x_2^{-p_{-1}} \cdot \exp\left[-\sum_{k=0}^{\infty} \frac{p_k}{(k+1)} x_2^{k+1}\right]$$

$$= \exp\left[-f(a)\right] x_2^{-p_{-1}} \left[1 - \sum_{k=0}^{\infty} \frac{p_k}{(k+1)} x_2^{k+1} + \frac{1}{2!}\left(\sum_{k=0}^{\infty} \frac{p_k}{(k+1)} x_2^{k+1}\right)^2 + \cdots\right]$$

$$\ldots 2.14.8$$

The final series expansion of the exponential is convergent if the expansion of the coefficient $P(x)$ is convergent. The denominator of the equation 2.14.6 may be handled by writing

$$\left[x_2^{2\alpha}\left(\sum_{\lambda=0}^{\infty} a_\lambda x_2^\lambda\right)^2\right]^{-1} = x_2^{-2\alpha}\left(\sum_{\lambda=0}^{\infty} a_\lambda x_2^\lambda\right)^{-2}$$

$$= x_2^{-2\alpha} \sum_{\lambda=0}^{\infty} b_\lambda x_2^\lambda \quad \ldots 2.14.9$$

Neglecting constant factors that will be picked up by the requirement $W(a) = 1$, we get

$$y_2(x) = y_1(x) \int^x x_2^{-p_{-1}-2\alpha}\left(\sum_{\lambda=0}^{\infty} c_\lambda x_2^\lambda\right) dx_2 \quad \ldots 2.14.10$$

by eq. 2.14.5

$$-p_{-1}-2\alpha = -n-1 \qquad n = \text{integer}$$

and $\therefore\ x_2^{-p_1-2\alpha} = x_2^{-n-1}$

$$y_2(x) = y_1(x)\int^x \left[c_0 x_2^{-n-1} + c_1 x_2^{-n} + c_2 x_2^{-n+1} + \dots + c_n x_2^{-1} + \dots\right] dx_2 \qquad \dots 2.14.11$$

The integration indicated by 2.14.11 consists of two parts

i) A power series starting with x^{-n}

ii) A logarithmic term from the integration of x^{-1} when $\lambda = n$. ~~then~~ This term always appears when n is an integer, unless c_n fortuitously vanish.

2.15 Example

second solⁿ of Bessel's eqⁿ.

Bessel's eqⁿ

$$x^2y'' + xy' + (x^2-n^2)y = 0$$

or $$y'' + \frac{1}{x}y' + \left(1 - \frac{n^2}{x^2}\right)y = 0 \quad \dots \quad (2.15.1)$$

Here $P(x) = 1/x$

$Q(x) = 1 - n^2/x^2 = 1$ for $n = 0$

Here $p_{-1} = 1$, $q_0 = 1$

all other p's & q's vanish

Thus the inditial eqⁿ is [from eq. 2.14.3]

$$k^2 + k(p_{-1}-1) = 0$$

or $k^2 = 0$

$\therefore k = 0$

The 1st soln of Bessel's original eqⁿ [eqⁿ 2.10.8] is

$$y(x) = a_0 2^n n!\, J_n(x) \quad \dots \quad (2.10.8)$$

Here $y_1(x) = J_0(x)$
$[n=0]$

with $$J_0(x) = \sum_{j=0}^{\infty} (-1)^j \frac{1}{(j!)^2}\left(\frac{x}{2}\right)^{2j} \quad \dots \quad (2.15.2)$$

$$= 1 - \frac{x^2}{4} + \frac{x^4}{64} - O(x^6) \dots \dots \quad (2.15.3)$$

To get the second sol^n from eq. 2.13.7

$$y_2(x) = y_1(x) \int^x dx_2 \frac{\exp\left[-\int^{x_2} P(x_1)dx_1\right]}{y_1^2(x_2)} \quad \ldots\ldots (2.13.7)$$

substituting for $y_1(x)$

$$y_2(x) = J_0(x) \int^x dx_2 \frac{\exp\left[-\int^{x_2} \frac{1}{x_1} dx_1\right]}{\left[1 - x_2^2/4 + \frac{x_2^3}{64} - \cdots\right]^2}$$

The numerator is

$$\Rightarrow \exp\left[-\int^{x_2} \frac{1}{x_1} dx_1\right] = \exp\left[-\ln x_2\right] = \frac{1}{x_2}$$

The denominator is by binomial expansion

$$\Rightarrow \left[1 - \frac{x_2^2}{4} + \frac{x_2^4}{64}\right]^{-2} = \left[1 + \frac{x_2^2}{2} + \frac{5}{32} x_2^4 + \cdots\right]$$

$$\therefore y_2(x) = J_0(x) \int^x dx_2 \left[\frac{1}{x_2} + \frac{x_2}{2} + \frac{5}{32} x_2^3 + \cdots\right]$$

$$= J_0(x) \left[\ln x + \frac{x^2}{4} + \frac{5}{128} x^4 + \cdots\right]$$

$\ldots\ldots (2.15.4)$

This eq^n agrees with the general form of Bessel's sol^n apart from certain constants.

The second sol^n has a logarithmic term and it diverges at the origin and is called irregular sol^n.

$y_1(x) \Rightarrow$ regular sol^n

$y_2(x) \Rightarrow$ irregular sol^n

Note: The nonhomogeneous eq^n has a particular integral which may be obtained by the method of variation of parameters or by Green's f^n technique or by Laplace transform method.