4. INTEGRAL EQUATIONS:

In some physical problems unknown functions say $y(x)$ appears under the integral sign. Such an equation is called an integral equation. Many integral equations can not be solved by standard techniques (to be discussed) but can be solved numerically very efficiently by present day computers.

4.1. Integral Eq^n from a differential equation:

Consider the DE

$$y''(x) = f(x, y) \qquad \text{4.1.1}$$

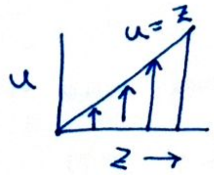
where $f(x,y)$ is any f^n of (x,y) but not of y'.
Eq^n 4.1.1 represents a large class of linear and non-linear 2nd order differential equations. We can convert 4.1.1 to an integral eq^n by integration

$$y'(x) = \int_0^x f(z, y)\, dz + c_1$$

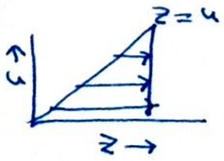
$$= \int_0^x f(z, y(z))\, dz + c_1$$

Again $$y(x) = \int_0^x du \int_0^u dz\, f(z, y(z)) + c_1 x + c_2$$

we ~~ch~~ can change the order of integration in the same region of uz plane which changes the integration limits.



$$\leftarrow \int_0^x du \int_0^u f(z)\, dz \quad \Rightarrow \quad \int_0^x dz\, f(z) \int_{z(=u)}^x du$$



changing the integration limits

$$y(x) = \int_0^x dz\, f(z, y(z)) \int_z^x du + c_1 x + c_2$$

$$= \int_0^x (x-z)\, f(z, y(z))\, dz + c_1 x + c_2$$

For general $f(x,y)$ this is called a nonlinear Volterra integral equation.

The boundary conditions imposed on $y(x)$ can be incorporated by fixing the constants c_1 and c_2

Eq. Let $y(0) = a$ and $y'(0) = 0$

we have $c_2 = a$; $c_1 = b$

Integral eq^n can be classified into two ways.

1) If the limits of the integration are fixed it is called a Fredholm eq^n; if one limit is variable it is called a Volterra eq^n.

2) If the unknown f^n appears only under the integral sign we label it as eq^n of the "first kind".

If it appears both inside and outside the integral it is called eqn of the "second kind".

4.2

Ex. Momentum Representation in Q.M.

The Schrödinger eqn is given by

$$H\Psi = E\Psi$$

where $H = -\frac{\hbar^2}{2m}\sum_i \nabla_i^2 - \sum_i \frac{Ze^2}{r_i} + \sum_{i<j} \frac{e^2}{r_{ij}}$

$$= -\frac{\hbar^2}{2m}\sum_i \nabla_i^2 - \sum V_i(\underline{r}) + \sum_{i<j}\frac{e^2}{r_{ij}}$$

For one particle system

$$H = -\frac{\hbar^2}{2m}\nabla^2 + V(\underline{r})$$

The sch. eqn is

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(r) - E\right]\Psi(r) = 0$$

or $(-\nabla^2 + a^2)\Psi(\underline{r}) = v(\underline{r})\Psi(\underline{r})$

where $a^2 = -\frac{2m}{\hbar^2}E$ and $v(\underline{r}) = -\frac{2m}{\hbar^2}V(\underline{r})$

for $v(\underline{r}, \underline{r}') = v(\underline{r}')\,\delta(\underline{r} - \underline{r}') \Rightarrow$ non local form

we have $(-\nabla^2 + a^2)\Psi(\underline{r}) = \int v(\underline{r}, \underline{r}')\psi(\underline{r}')\,d\underline{r}'$

we now make a F.T.

$$\int[-\nabla^2 + a^2]\psi(\underline{r})\,e^{-i\underline{k}.\underline{r}}\,d\underline{r}$$

$$= \iint v(\underline{r}, \underline{r}')\,e^{-i\underline{k}.\underline{r}}\,\psi(\underline{r}')\,d\underline{r}\,d\underline{r}'$$

$$\boxed{\begin{aligned}\Phi(\underline{k}) &= \frac{1}{(2\pi)^{3/2}}\int e^{-i\underline{k}.\underline{r}}\psi(\underline{r})\,d\underline{r} \\ \psi(\underline{r}) &= \frac{1}{(2\pi)^{3/2}}\int e^{i\underline{k}'.\underline{r}}\Phi(\underline{k}')\,d\underline{k}'\end{aligned}}$$

we write

$\underline{p} = \hbar\underline{k} \Rightarrow$ momentum

The last eqn gives

$$\int (k^2 + a^2)\psi(\underline{r})\,e^{-i\underline{k}.\underline{r}}\,d\underline{r}$$

This can be obtained by int. by parts & applying boundary cond. on the wave fn.

$$= \iint v(\underline{r}, \underline{r}')\psi(\underline{r}')\,e^{-i\underline{k}.\underline{r}}\,d\underline{r}'d\underline{r}$$

or $(2\pi)^{3/2}\left[+k^2+a^2\right]\Phi(k)$

$$= \frac{1}{(2\pi)^{3/2}}\iiint v(\underline{r},\underline{r}')\,\Phi(\underline{k}')\,e^{-i(\underline{k}\cdot\underline{r}-\underline{k}'\cdot\underline{r}')}\,d\underline{r}'\,d\underline{r}\,d\underline{k}'$$

write $f(\underline{k},\underline{k}') = \frac{1}{(2\pi)^3}\iint v(\underline{r},\underline{r}')\,e^{-i(\underline{k}\cdot\underline{r}-\underline{k}'\cdot\underline{r}')}\,dr'\,dr$

Then the eq^n looks like

$$\left[k^2+a^2\right]\Phi(\underline{k}) = \int f(\underline{k},\underline{k}')\,\Phi(k')\,d\underline{k}'$$

This is a homogeneous Fredholm eq^n of the second kind in which the parameter a^2 corresponds to the eigenvalue. For local interactions we have

$$f(\underline{k},\underline{k}') = f(\underline{k}-\underline{k}')$$

This is the momentum representation equivalent to an ordinary static interaction potential in ordinary space. The momentum f^n $\Phi(\underline{k})$ satisfies an integral eq^n [Last one]. We assumed that the F.T. exists. For an oscillator potential $V(r)\sim r^2$, the integral does not exist and we get divergent oscillations.

4.3. Transformation of a DE into an I.E:

consider a second order linear DE

$$y'' + A(x)y' + B(x)y = g(x) \quad \cdots\cdots \quad 4.3.1$$

with $y(a)=y_0$; $y'(a)=y_0'$

we wish to transform it into an integral eq^n I.E.

From 4.3.1

$$y' + \int_a^x A(x)y'\,dx + \int_a^x B(x)\,y\,dx = \int_a^x g(x)\,dx + y_0'$$

Integrating by parts.

or $$y' = -Ay - \int_a^x (B-A')y\,dx + \int_a^x g(x)\,dx + y_0' + A(a)y_0$$

The initial ~~so~~ conditions are being absorbed in new eqn

Integrating again

$$y = -\int_a^x Ay\,dx - \int_a^x\int_a^x [B(t) - A'(t)]y(t)\,dt\,dx$$

$$+ \int_a^x\int_a^x g(t)\,dt\,dx + [A(a)y_0 + y_0'](x-a) + y_0 \quad \ldots\ldots (4.3.2)$$

To transform this eqn in a more neat form we use the relation

$$\int_a^x\int_a^x f(t)\,dt\,dx = \int_a^x (x-t)f(t)\,dt \quad \ldots\ldots (4.3.3)$$

To have a proof, Differentiate both sides w.r.t x we have

$$L.S = \int_a^x f(t)\,dt$$

$$R.S = \int_a^x f(t)\,dt$$

since the derivatives are equal the original expression may differ by a constant which $\to 0$ as $x \to a$

Applying it to eqn 4.3.2

$$y(x) = -\int_a^x \{A(t) + (x-t)(B(t) - A'(t))\}y(t)\,dt$$

$$+ \int_a^x (x-t)g(t)\,dt + \{A(a)y_0 + y_0'\}(x-a) + y_0 \quad \ldots\ldots 4.3.4$$

Let us denote

$$K(x,t) = (t-x)[B(t) - A'(t)] - A(t)$$

and $$f(x) = \int_a^x (x-t)g(t)\,dt + [A(a)y_0 + y_0'](x-a) + y_0$$

then 4.3.4 becomes

$$y(x) = f(x) + \int_a^x k(x,t)\, y(t)\, dt \quad \ldots\ldots 4.3.5.$$

which is a volterra eqn of the second kind.

Ex. Linear Oscillator

Consider $y'' + \omega^2 y = 0$ - - - - - - - 4.3.6

with $y(0) = 0$

$y'(0) = 1$

This shows that $A(x) = 0$; $B(x) = \omega^2$; $g(x) = 0$

substituting in eqn 4.3.4

$$y(x) = x + \omega^2 \int_0^x (t-x)\, y(t)\, dt \quad \ldots\ldots 4.3.7$$

The soln is given by

$$y(x) = \frac{1}{\omega} \sin \omega x$$

Let us now consider the eqn with the initial conditions

$y(0) = 0$

$y(b) = 0$

Since $y'(0)$ is not given we have to modify the procedure

$$y'' + \omega^2 y = 0$$

Integrating

$$y' = -\omega^2 \int_0^x y\, dx + y'(0) \quad \ldots\ldots (4.3.8)$$

Integrating again

$$y = -\omega^2 \int_0^x (x-t)\, y(t)\, dt + y'(0)\, x$$

To eliminate $y'(0)$ we have to apply $y(b) = 0$

This gives

$$0 = -\omega^2 \int_0^b (b-t)\, y(t)\, dt + b y'(0)$$

or $$b y'(0) = \omega^2 \int_0^b (b-t)\, y(t)\, dt \quad \ldots\ldots \quad 4.3.9$$

substituting in 4.3.8 we get

$$y(x) = -\omega^2 \int_0^x (x-t)\, y(t)\, dt + \frac{\omega^2 x}{b} \int_0^b (b-t)\, y(t)\, dt$$

Now $\int_0^b \Rightarrow \int_0^x + \int_x^b$

$$\therefore y(x) = -\omega^2 \int_0^x \left[(x-t)\, y(t) - \frac{x}{b}(b-t)\, y(t)\right] dt$$

$$+ \frac{\omega^2 x}{b} \int_x^b (b-t)\, y(t)\, dt$$

$$= +\omega^2 \int_0^x \frac{t}{b}(b-x)\, y(t)\, dt + \omega^2 \int_x^b \frac{x}{b}(b-t)\, y(t)\, dt$$

$$\left[\because\ x-t-\frac{x}{b}(b-t) = \frac{t}{b}(x-b)\right]$$

If we define a kernel

$$k(x,t) = \frac{t}{b}(b-x) \qquad t<x$$

$$= \frac{x}{b}(b-t) \qquad t>x$$

Then we have

$$y(x) = \omega^2 \int_0^b k(x,t)\, y(t)\, dt \quad \ldots\ldots \quad 4.3.10$$

This is homogeneous Fredholm eqn of second kind.

<u>Note:</u> In the transformation of linear second order DE into a IE, the boundary conditions play a decisive role. If we have initial conditions only one end of the interval, the DE transforms into a volterra eqn. For the case of linear oscillator with initial conditions on both ends of the interval, the DE transforms into a Fredholm IE. Also it is to be noted that always the back transformation of an IE into a DE is not possible. There exists IE for which no DE is known.

4.4.

There exists in general three techniques for solving integral eq^n.

1) Power series sol^n (due to Neumann, Liouville & Volterra)

2) Method of separation or degenerate kernels.

3) Integral transform method.

In addition complicated eq^{n}s can be solved by numerical methods.

1) Neumann Series sol^n:-

Consider the eq^n

$$y(x) = f(x) + \lambda \int_a^b k(x,z)\, y(z)\, dz \quad \cdots\cdots\cdots (4.4.1)$$

This is Fredholm's eq^n of the 2nd kind if a, b are constants or the upper limit is variable for a Volterra eq^n.

If λ is small then a crude approximation (may not work) may be $y(x) \approx y_0(x) = f(x) \quad \cdots\cdots\cdots (4.4.2)$

Then $\quad y_1(x) = f(x) + \lambda \int_a^b k(x,z) f(z) dz$

We repeat this procedure

$$y_2(x) = f(x) + \lambda \int_a^b k(x,z)\, y_1(z)\, dz$$

$$= f(x) + \lambda \int_a^b k(x,z_1) f(z_1) dz_1$$

$$+ \lambda^2 \int dz_1 \int_a^b dz_2\, K(x, z_1) k(z_1, z_2) f(z_2)$$

We repeat the procedure to get

$$y_n(x) = f(x) + \sum_{m=1}^{n} \lambda^m \int_a^b k_m(x,z) f(z) dz \quad \cdots\cdots (4.4.3)$$

Where $k_1(x,z) = K(x,z)$

$$k_2(x,z) = \int_a^b k(x,z_1) K(z_1, z) dz$$

$$k_3(x,z) = \int_a^b \int_a^b k(x,z_1) k(z_1 z_2) K(z_2, z)\, dz_1 dz_2$$

$\vdots$

$$k_n(x,z) = \int_a^b k(x,z_1)\, K_{n-1}(z_1, z)\, dz_1$$

$$= \int_a^b k(x,z_1) k(z_1, z_2) \cdots K(z_{n-1}, z) dz_1 dz_2 \ldots dz_{n-1}$$

The sol^n of the original eq^n is

$\lim_{n\to\infty} y_n(x)$ provided the series converges.

We may write $y_n(x) = \sum_{i=0}^{n} \lambda^i u_i(x)$ } This is not necessary

Another form where $u_0(x) = y_0(x) = f(x)$

$$u_1(x) = \int_a^b K(x,z_1) f(z_1)\,dz_1$$

$$u_2(x) = \int_a^b \int_a^b K(x,z_1) K(z_1, z_2) f(z_2)\,dz_2\,dz_1$$

$$\vdots$$

$$u_n(x) = \int_a^b \int_a^b \cdots \int_a^b K(x,z_1)K(z_1,z_2)\ldots K(z_{n-1},z_n)\, f(z_n)\,dz_n\,dz_{n-1} \ldots dz_1$$

with $y(x) = \lim_{n\to\infty} y_n(x) = \lim_{n\to\infty} \sum_{i=0}^{n} \lambda^i u_i(x)$ . . . (4.4.4)

The standard Cauchy ratio test for the convergence of the series may be checked with

This may also be written as

$$y(x) = f(x) + \lambda \int_a^b R(x,z,z)\, f(z)\,dz \quad . \; . \; . \; . \quad (4.4.5)$$

where the resolvent kernel $R(x,z,z) = \sum_{n=0}^{\infty} \lambda_m^m K_{m+1}(x,z)$

Ex. 1

solve $\quad y(x) = x + \lambda \int_0^1 xz\, y(z)\,dz$ . . . . . (4.4.6)

Let $y(x) \approx y_0(x) \approx x$ as the first approx.

$$y_1(x) = x + \lambda \int_0^1 xz\, y_0(z)\,dz + x + \lambda \int_0^1 x z^2\,dz$$

$$= x + \frac{\lambda x}{3}$$

$$\therefore y_2 = x + \lambda \int_0^1 xz\, y_1(z)\,dz$$

$$= x + \lambda \int_0^1 xz\left(z + \frac{\lambda z}{3}\right)dz = x + \frac{\lambda x}{3} + \frac{\lambda^2 x}{9}$$

$$= x + \left(\frac{\lambda}{3}\right)x + \left(\frac{\lambda}{3}\right)^2 x$$

continuing in this manner we get

$$y(x) = x + \left[\frac{\lambda}{3} + \left(\frac{\lambda}{3}\right)^2 + \left(\frac{\lambda}{3}\right)^3 + \left(\frac{\lambda}{3}\right)^4 + \cdots\right]x$$

$$= x + \left(\frac{\lambda}{3}\right)\left[1 + \frac{\lambda}{3} + \left(\frac{\lambda}{3}\right)^2 + \cdots\right]x$$

This is a G.P series with 1st term 1 and $\frac{\lambda}{3}$ as common ratio.

Provided $|\lambda| < 3$ we get

$$y(x) = x + \frac{\lambda}{3}x \frac{1}{1-\frac{\lambda}{3}} = x + x\frac{\lambda}{3-\lambda}$$

or $y(x) = \dfrac{3x}{3-\lambda}$

There is a more elegant method due to Fredholm where the resolvent Kernel is expressed in terms of ratio of two infinite series.

Ex. 2

Consider the I.E.

$$y(x) = x + \frac{1}{2}\int_{-1}^{1}(z-x)\,y(z)\,dz$$

we take

$$y_0(x) = x$$

$$\therefore\ y_1(x) = x + \frac{1}{2}\int_{-1}^{1}(z-x)\,y_0(z)\,dz$$

$$= x + \frac{1}{2}\int_{-1}^{1}(z-x)\,z\,dz$$

$$= x + \frac{1}{2}\int_{-1}^{1}(z^2 - xz)\,dz$$

$$= x + \frac{1}{2}\left[\frac{1}{3}z^3 - \frac{1}{2}xz^2\right]_{-1}^{1}$$

$$= x + \frac{1}{3}$$ [The even function term vanishes when limits are substituted.]

$$\therefore\ y_2(x) = x + \frac{1}{2}\int_{-1}^{1}(z-x)\left(z + \frac{1}{3}\right)dz$$

$$= x + \frac{1}{2}\int_{-1}^{1}(z-x)\,z\,dz + \frac{1}{2}\int_{-1}^{1}(z-x)\frac{1}{3}\,dz$$

$$= x + \frac{1}{3} - \frac{x}{3}$$

and so on

$$y_3(x) = x + \frac{1}{3} - \frac{x}{3} - \left(\frac{1}{3}\right)^2$$

By induction $y_{2n}(x) = x + \sum_{m=1}^{n}(-1)^{m-1}3^{-m} - x\sum_{m=1}^{n}(-1)^{m-1}3^{-m}$

when $n \to \infty$ $y_{2n}(x) = x + (1-x)\sum_{m=1}^{n}(-1)^{m-1}3^{-m}$

$$= x + (1-x)\frac{1}{3}\cdot\left(1+\frac{1}{3}\right)^{-1}$$

$$\left[\left(1+\frac{1}{3}\right)^{-1} = 1 - \frac{1}{3} + \frac{1}{3^2} - \frac{1}{3^3} + \frac{1}{3^4} - \cdots\right] = x + \frac{1-x}{4} = \frac{3}{4}x + \frac{1}{4}$$

$$= \left(1-\frac{1}{3}\right)\left(1 + \frac{1}{3^2} + \frac{1}{3^4} + \cdots\right]$$

$$= \frac{3}{4}\ \ ]$$

4.5. Method of Separable Kernel :-

[Most straight forward method for Fredholm's eqns]

We have the eq^n

$$y(x) = f(x) + \lambda \int_a^b k(x,z)\, y(z)\, dz \quad \ldots\ldots\ldots \quad 4.5.1$$

Let $$k(x,z) = \sum_{i=1}^{n} \phi_i(x) \psi_i(z) \quad \ldots\ldots\ldots \quad 4.5.2$$

Where the functions $\phi_i(x)$ & $\psi_i(z)$ are f^{n}s of x and z only and the summation extends over a finite range.

substitute 4.5.2 in 4.5.1 and

$$y(x) = f(x) + \lambda \int_a^b \sum_{i=1}^{n} \phi_i(x) \psi_i(z) y(z)\, dz$$

$$= f(x) + \lambda \sum_{i=1}^{n} \phi_i(x) \int_a^b \psi_i(z) y(z)\, dz$$

For a Fredholm eq^n the limits are constants and

$$\int_a^b \psi_i(z)\, y(z)\, dz = c_i = \text{Const.} \quad \ldots\ldots\ldots \quad 4.5.3.$$

Hence

$$y(x) = f(x) + \lambda \sum_{i=1}^{n} \phi_i(x) c_i = f(x) + \lambda \sum_{i=1}^{n} c_i \phi_i(x) \quad \ldots \quad 4.5.4$$

Eqn 4.5.3 gives the required c_i's.
Now the x dependence of c_i in eq^n 4.5.4 may be eliminated by the following procedure.

From eq^n 4.5.4

$$\int_a^b \psi_i(x) y(x)\, dx = \int_a^b \psi_i(x) f(x)\, dx + \lambda \sum_{j=1}^{n} c_j \psi_i(x)\, \phi_j(x)\, dx$$

or $$c_i = b_i + \lambda \sum_{j=1}^{n} a_{ij}\, c_j \quad \ldots\ldots\ldots \quad 4.5.5$$

where $$b_i = \int_a^b \psi_i(x) f(x)\, dx$$

$$a_{ij} = \int_a^b \psi_i(x) \phi_j(x)\, dx \quad \ldots\ldots\ldots \quad 4.5.5.a$$

This is a matrix eq^n

$$\underline{c} = \underline{b} + \lambda \underline{\underline{A}}\, \underline{c}$$

$$\left.\begin{array}{l} \text{or } \underline{\underline{b}} = (1 - \lambda \underline{\underline{A}})\underline{c} \\ \text{or } \underline{c} = (1 - \lambda \underline{\underline{A}})^{-1} \underline{b} \end{array}\right\} \quad \ldots\ldots \quad 4.5.6$$

Eq^n 4.5.6 is equivalent to a set of simultaneous linear algebraic equations.

$$(1-\lambda a_{11})c_1 - \lambda a_{12}c_2 - \ldots\ldots \lambda a_{1n}c_n = b_1$$

$$\lambda a_{21}c_1 + (1-\lambda a_{22})c_2 - \ldots\ldots -\lambda a_{2n}c_n = b_2$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\lambda a_{n1}c_1 - \lambda a_{n2}c_2 - \ldots\ldots + (1-\lambda a_{nn})c_n = b_n$$

If the eq^n 4.5.1 is homogeneous $f(x) = 0$, $\underline{b} = 0$

To get nontrivial $\underline{c}$'s we must have the det

$$|(1-\lambda A)| = 0 \quad \ldots\ldots\ldots\ldots \quad 4.5.7.$$

The roots of 4.5.7 yield the eigenvalues and $(1-\lambda \underline{\underline{A}})\underline{c} = 0$ gives the c's. and the soln is given by eq^n 4.5.4.

Ex.

Consider a simple case of homogeneous Fredholm's eq^n

$$y(x) = \lambda \int_{-1}^{1} (z+x)\, y(z)\, dz \quad \ldots\ldots\ldots \quad 4.5.8$$

Here $K(x,z) = (z+x) = \sum_i \phi_i(x)\psi_i(z)$

$$\therefore \phi_1 = 1 \; ; \; \psi_1 = z$$

$$\phi_2 = x \; ; \; \psi_2 = 1$$

we have by 4.5.5

$$c_i = b_i + \lambda \sum_{j=1}^{n} a_{ij} c_j$$

Here b_i's are zero

And eq^n 4.5.5a yields

$$a_{11} = \int_{-1}^{1} \psi_1 \phi_1 dx = \int_{-1}^{1} x\, dx = 0$$

$$a_{22} = \int_{-1}^{1} \psi_2 \phi_2 dx = \int_{-1}^{1} x dx = 0$$

$$a_{12} = \int_{-1}^{1} \psi_1 \phi_2 dx = \int_{-1}^{1} x^2 dx = \frac{2}{3}$$

$$a_{21} = \int_{-1}^{1} \psi_2 \phi_1 dx = \int_{-1}^{1} 1.dx = 2$$

Thus the secular eqn is $|I - \lambda \underline{A}| = 0$

or $\begin{vmatrix} 1 - & \lambda \times \frac{2}{3} \\ -\lambda \times 2 & 1 \end{vmatrix} = 0$ or $1 - \frac{4\lambda^2}{3} = 0 \Rightarrow \lambda = \pm \frac{\sqrt{3}}{2}$

again $\begin{pmatrix} 1 - & \frac{2}{3}\lambda \\ -2\lambda & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0 \Rightarrow c_1 - \frac{2\lambda}{3} c_2 = 0$

$$-2\lambda c_1 + c_2 = 0$$

These solutions are consistent with λ. we have $c_2 = \pm\sqrt{3}$

Thus by 4.5.6

$$c_1 \pm c_2/\sqrt{3} = 0$$

with a choice of $c_1 = 1$ we have $c_2 = \pm\sqrt{3}$

$$y(x) = \lambda(c_1 \phi_1 + c_2 \phi_2)$$

$$y_1(x) = \frac{\sqrt{3}}{2}(1 + \sqrt{3}x) \qquad \lambda = \sqrt{3}/2$$

$$y_2(x) = -\frac{\sqrt{3}}{2}(1 - \sqrt{3}x) \qquad \lambda = -\sqrt{3}/2$$

Since our eqn is homogeneous the normalisation of $y(x)$ is arbitrary.

Ex. solve $y(x) = x + \lambda \int_0^1 (xz + z^2) y(z) dz$ 4.5.9

Here $k(x, z) = xz + z^2 \Rightarrow$ This is separable

Here we have $\phi_1(x) = x, \ \phi_2(x) = 1$

$\psi_1(z) = z, \ \psi_2(z) = z^2$

$\therefore$ from 4.5.4

$$y(x) = x + \lambda(c_1 x + c_2)$$

where $c_1 = \int_0^1 \psi_1(z) y(z) dz = \int_0^1 z\{z + \lambda(c_1 z + c_2)\} dz$

$$= \frac{1}{3} + \frac{1}{3}\lambda c_1 + \frac{1}{2}\lambda c_2$$

or $c_1(1-\frac{1}{3}\lambda) - \frac{1}{2}\lambda c_2 = 1/3$

And $c_2 = \int_0^1 \psi_2(z)\, y(z)dz = \int_0^1 z^2\{z + \lambda[c_1 z + c_2]\}\,dz$

$$= \frac{1}{4} + \frac{1}{4}\lambda c_1 + \frac{1}{3}\lambda c_2$$

$$\therefore \frac{1}{4}\lambda c_1 + (\frac{1}{3}\lambda - 1)c_2 = -\frac{1}{4}$$

Here we have the eqns.

$$a_{11}c_1 + a_{12}c_2 = b_1$$
$$a_{21}c_1 + a_{22}c_2 = b_2$$

where $a_{11} = (1-\frac{1}{3}\lambda)$; $a_{12} = -\frac{1}{2}\lambda$; $b_1 = 1/3$

$a_{21} = \frac{1}{4}\lambda$; $a_{22} = \frac{1}{3}\lambda - 1$; $b_2 = -1/4$

By Cramer's rule

$$c_1 = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix} \Big/ \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \; ; \quad c_2 = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix} \Big/ \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = (1-\frac{1}{3}\lambda)(\frac{1}{3}\lambda - 1) + \frac{1}{2}\lambda \cdot \frac{1}{4}\lambda$$

$$= -(\frac{1}{9}\lambda^2 - \frac{2}{3}\lambda + 1) + \frac{1}{8}\lambda^2$$

$$= \frac{\lambda^2 + 48\lambda - 72}{72}$$

Now $\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix} = b_1 a_{22} - b_2 a_{12} = \frac{1}{3}(\frac{1}{3}\lambda - 1) + \frac{1}{4}(-\frac{1}{2}\lambda)$

$$= \frac{\lambda}{9} - \frac{1}{3} - \frac{1}{8}\lambda = \frac{-\lambda - 24}{72} = -\frac{(\lambda+24)}{72}$$

$$\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix} = a_{11}b_2 - b_1 a_{21} = (1-\frac{1}{3}\lambda)(-\frac{1}{4}) - \frac{1}{3}\frac{1}{4}\lambda = -\frac{1}{4} + \frac{\lambda}{12} - \frac{\lambda}{12}$$

$$= -\frac{1}{4}$$

Hence $c_1 = \dfrac{-(\lambda+24)/72}{\dfrac{(\lambda^2+48\lambda-72)}{72}} = \dfrac{\lambda+24}{72-48\lambda-\lambda^2}$

$$c_2 = \frac{-\frac{1}{4}}{(\lambda^2+48\lambda-72)/72} = \frac{18}{72-48\lambda-\lambda^2}$$

which yields

$$y(x) = \frac{(72-24\lambda)x + 18\lambda}{72-48\lambda-\lambda^2}$$

Ex:-

Consider the homogeneous Fredholm's eqⁿ

$$y(x) = \lambda \int_0^{\pi} \sin(x+z)\, y(z)\, dz$$

Here $k(x,z) = \sin(x+z) = \sin x \cos z + \cos x \sin z$

Thus from 4.5.2

$$\phi_1(x) = \sin x \; ; \quad \psi_1(z) = \cos z$$

$$\phi_2(x) = \cos x \; ; \quad \psi_2(z) = \sin z$$

The soln according to 4.5.4 is

$$y(x) = \lambda (c_1 \sin x + c_2 \cos x)$$

with $c_1 = \int_0^{\pi} \cos z \, y(z)\, dz$

$$= \int_0^{\pi} \cos z \left[\lambda c_1 \sin z + \lambda c_2 \cos z \right] dz$$

$$= \lambda c_1 \int_0^{\pi} \sin z \cos z \, dz + \lambda c_2 \int_0^{\pi} \cos^2 z \, dz$$

$$= +\lambda c_1 \int_0^{\pi} \sin z \, d(\sin z) + \lambda c_2 \int_0^{\pi} \frac{1}{2} (1 + \cos 2z)\, dz$$

The trigonometric integrals vanish and we have

$$c_1 = \lambda \frac{\pi}{2} c_2$$

similarly

$$c_2 = \int_0^{\pi} \sin z \left[\lambda c_1 \sin z + \lambda c_2 \cos z \right] dz$$

$$= \lambda c_1 \int_0^{\pi} \sin^2 z \, dz + \lambda c_2 \int_0^{\pi} \sin z \cos z \, dz$$

$$= \frac{\lambda \pi}{2} c_1$$

The two relations yield $c_1 = \frac{\lambda^2 \pi^2}{4} c_1$

Assuming $c_1 \neq 0$ we get $\lambda^2 = \frac{4}{\pi^2}$ or $\lambda = \pm \frac{2}{\pi}$

the two eigen values of the integral eqⁿ.

substituting the values in the soln we have

$$y_1(x) = A(\sin x + \cos x)$$

$$y_2(x) = B(\sin x - \cos x)$$

where A and B are arbitrary constants.

4.6. Method of Differentiation :-

A closed form solution to a volterra eqⁿ may sometimes be obtained by differentiating the eqn to obtain the corresponding DE which may be easier to solve.

consider $y(x) = x - \int_0^x x z^2 y(z)\,dz$

or $\frac{y(x)}{x} = 1 - \int_0^x z^2 y(z)\,dz$

$\therefore \frac{d}{dx}\left[\frac{y(x)}{x}\right] = -x^2 y(x) = -x^3\left[\frac{y(x)}{x}\right]$

By straightforward integration

$$\ln\left[\frac{y(x)}{x}\right] = -\int x^3 dx + c$$

$$= -\frac{1}{4}x^4 + c$$

Thus $y(x) = x \exp\left[-\frac{x^4}{4} + c\right]$

$$= Ax \exp\left[-\frac{x^4}{4}\right]$$

where A is some arbitrary constant.

Since the original eqⁿ does not have a constant, the solⁿ should not have any. By substitution one finds $A=1$.

4.7. Integral Transform Method :-

If the kernel of an I.E. can be written as a function of the difference $(x-z)$ of its two arguments then it is called a displacement kernel. An integral eqⁿ having such a kernel and which also has the limits $-\infty$ to $+\infty$ may be solved by the use of Fourier transforms.

Fourier Transforms:

The fourier transform of a function $f(t)$ is defined by

$$f(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\,dt \quad \dots\dots 4.7.1.$$

Inverse F. Transform is

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\omega)\, e^{i\omega t}\, d\omega \quad \cdots\cdots \quad 4.7.2$$

[In three dimensional case

$$f(\underline{k}) = \frac{1}{(2\pi)^{3/2}} \int f(\underline{r})\, e^{-i\underline{k}\cdot\bar{r}}\, d\underline{r} \quad \cdots\cdots \quad 4.7.3.$$

$$f(\underline{r}) = \frac{1}{(2\pi)^{3/2}} \int f(\underline{k})\, e^{i\bar{k}\cdot\bar{r}}\, d\underline{k} \quad] \cdots\cdots \quad 4.7.4$$

EX. F.T. of $f(t) = Ae^{-\lambda t}$ for $t \geq 0$, $\lambda > 0$
$= 0$ for $t < 0$

$$f(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, \cancel{dt}\, e^{i\omega t}\, dt = \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} e^{-i\omega t} f(t)\, dt + \int_{0}^{\infty} e^{-i\omega t} f(t)\, dt \right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-i\omega t} f(t)\, dt = \frac{1}{\sqrt{2\pi}} \cdot A \int_{0}^{\infty} e^{-(i\omega+\lambda)t}\, dt$$

$$= \frac{A}{\sqrt{2\pi}} \frac{1}{-(i\omega+\lambda)} \left[e^{-(i\omega+\lambda)t} \right]_{0}^{\infty}$$

$$= \frac{A}{\sqrt{2\pi}} \frac{1}{(\lambda + i\omega)}$$

To go into details we have to know the Fourier inversion theorem which states

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, e^{i\omega t} \int_{-\infty}^{\infty} du\, f(u)\, e^{-i\omega u} \quad \cdots\cdots \quad 4.7.5$$

This is obtained from Fourier's theorem which states that any periodic function ~~which~~ with period T may be represented in a complex fourier series as

$$f(t) = \sum_{n=-\infty}^{\infty} c_n\, e^{+2\pi i n t/T} = \sum_{n=-\infty}^{\infty} c_n\, e^{+i\omega_n t} \qquad \omega_n = \frac{2\pi n}{T}$$

As $T \to \infty$, $\omega_n = \frac{2\pi n}{T} \to 0$

and $\Delta\omega = \omega_n - \omega_{n-1} = \frac{2\pi}{T} \to 0$

The spectrum of allowed frequencies ω_n tending ~~to~~ to continuum, the Fourier sum may be represented by an integral. The coefficients c_n becomes functions of the continuous variable ω as follows

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{-i2\pi n t/T}\, dt$$

$$= \left(\frac{\Delta\omega}{2\pi}\right)\int_{-T/2}^{T/2} f(t)\, e^{-i\omega_n t}\, dt$$

Thus $$f(t) = \sum_{n=-\infty}^{\infty}\left(\frac{\Delta\omega}{2\pi}\right)\int_{-T/2}^{T/2} f(u)\, e^{-i\omega_n u}\, du\, e^{i\omega_n t}$$

Here $\omega_n = 2\pi n/T \rightarrow$ still discrete

From the mathematical definition of an integral

$$\sum_{n=-\infty}^{\infty}\left(\frac{\Delta\omega}{2\pi}\right) g(\omega_n)\, e^{i\omega_n t} \rightarrow \frac{1}{(2\pi)}\int_{-\infty} g(\omega)\, e^{i\omega t}\, d\omega$$

Here $$g(\omega_n) = \int_{-T/2}^{T/2} f(u)\, e^{-i\omega_n u}\, du$$

Hence
$$f(t) = \frac{1}{(2\pi)}\int_{-\infty}^{\infty} d\omega\, e^{i\omega t}\int_{-\infty}^{\infty} du\, e^{-i\omega u} f(u)\, du$$

$\Rightarrow$ Fourier inversion theorem.

we write
$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\int_{-\infty}^{\infty} e^{i\omega(t-u)} f(u)\, du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\left[\int_{-\infty}^{\infty} e^{i\omega(t-u)}\, d\omega\right] du$$

$$\equiv \int_{-\infty}^{\infty} f(u)\, \delta(t-u)\, du$$

where $\delta(t-u) \Rightarrow$ Dirac delta f^n

Thus $$\delta(t-u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(t-u)}\, d\omega$$

$$\delta(\underline{r}_1 - \underline{r}_2) = \delta(x_1 - x_2)\,\delta(y_1 - y_2)\,\delta(z_1 - z_2)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik_1(x_1-x_2)}\, dk_1 \left(\frac{1}{2\pi}\right)\int_{-\infty}^{\infty} e^{ik_2(y_1-y_2)}\, dk_2$$

$$* \left(\frac{1}{2\pi}\right)\int_{-\infty}^{\infty} e^{ik_3(z_1-z_2)}\, dk_3$$

$$= \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} e^{i\underline{k}\cdot(\underline{r}_1 - \underline{r}_2)}\, d\underline{k}$$

A point charge q at a point $\underline{r}_0$ may be represented by a density f^n

$$\rho(\underline{r}) = q\,\delta(\underline{r}-\underline{r}_0)$$

$$\int_V \rho(\underline{r})\,dv = q \int_V \delta(\underline{r}-\underline{r}_0)\,dv = q \quad \text{if } \underline{r}_0 \text{ is inside } V$$
$$= 0 \quad \text{otherwise.}$$

We now write the Heaviside f^n or unit step f^n defined by $H(t) = \begin{cases} 1 & t>0 \\ 0 & t<0 \end{cases}$

The f^n is discontinuous at $t=0$ and usually $H(0)$ is taken as $1/2$

$H(t)$ is related to $\delta(t)$ by

$$H'(t) = \delta(t)$$

Proof: $$\int_{-\infty}^{\infty} f(t)H'(t)\,dt = f(t)H(t)\Big]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(t)\cdot H(t)\,dt$$

$$= f(\infty) - \int_0^{\infty} f'(t)\,dt$$

$$= f(\infty) - \left[f(t)\right]_0^{\infty}$$

$$= f(0)$$

we know that

$$f(x) = \int_{-\infty}^{\infty} f(t)\,\delta(x-t)\,dt$$

$$= \int_{-\infty}^{\infty} f(t)\,\delta(t-x)\,dt$$

$$[\because \delta(x) = \delta(-x)]$$

Hence putting $x=0$ we have

$$H'(t) = \delta(t)$$

4.8 Fourier Convolution Theorem :-

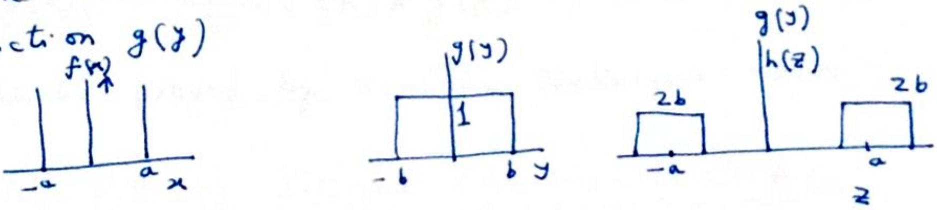
Let us consider two functions $f(x)$ and $g(x)$ with Fourier transforms $f(t)$ and $g(t)$

The quantity

$$f*g = h(z) = \int_{-\infty}^{\infty} f(x)\, g(z-x)\, dx \quad \cdots\cdots \quad 4.8.1$$

is called the convolution of the f^n f and g. The convolution defined above is commutative $(f*g = g*f)$, associative and distributive. It is called "Faltung" theorem (meaning folding)

Ex. Find the convolution of $f(x) = \delta(x+a) + \delta(x-a)$ with the function $g(y)$

$f(x) =$ 

$$h(z) = \int_{-\infty}^{\infty} f(x)\, g(z-x)\, dx = \int_{-\infty}^{\infty} \delta(x+a)\, g(z-x)\, dx + \int_{-\infty}^{\infty} \delta(x-a)\, g(z-x)\, dx$$

$$= g(z+a) + g(z-a) \Rightarrow \text{shown in fig with } g(y) \text{ as shown.}$$

Let us consider the Fourier transform of the convolution $f*g$

$$h(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(z)\, e^{-ikz}\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dz\, e^{-ikz} \int_{-\infty}^{\infty} f(x)\, g(z-x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, f(x) \int_{-\infty}^{\infty} g(z-x)\, e^{-ikz}\, dz$$

Let $z - x = u$

then $h(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, f(x) \int_{-\infty}^{\infty} g(u)\, e^{-ik(u+x)}\, du$

$$= \frac{1}{\sqrt{2\pi}}\cancel{\sqrt{2\pi}\, f(k)} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, f(x) e^{-ikx} \int_{-\infty}^{\infty} g(u) e^{-iku} du$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \sqrt{2\pi}\, f(k) \sqrt{2\pi}\, g(k)$$

$$= \sqrt{2\pi}\, f(k) * g(k) \quad \cdots\cdots\cdots 4.8.2$$

Hence F.T of the convolution $f*g$ is equal to the product of the separate F.T's multiplied by $\sqrt{2\pi}$. Eq^n 4.8.2 is called "Convolution Theorem". The converse is also true i.e. F.T of a product $f(x) g(x)$ is

$$\mathcal{F}\left[f(x) g(x)\right] = \frac{1}{\sqrt{2\pi}} f(k) * g(k)$$

This can be proved by similar technique above.

4.9 Soln of I.E by Integral Transform Method:

If the kernel of an integral equation can be written as a function of the difference $x-z$ of its two arguments then it is called a displacement kernel. As integral eq^n having such a kernel having the integration limits $-\infty$ to ∞ may be solved by the use of Fourier transforms

Let

$$y(x) = f(x) + \lambda \int_{-\infty}^{\infty} K(x-z) y(z) dz \quad \cdots\cdots 4.9.1$$

The integral over z takes the form of a convolution we take F.T of eq 4.9.1 then

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y(x) e^{-ikx} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$

$$+ \lambda \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, e^{-ikx} \int_{-\infty}^{\infty} K(x-z) y(z) dz$$

By convolution theorem we have

$$Y(k) = F(k) + \lambda\sqrt{2\pi}\, K(k)\, Y(k) \quad \cdots\cdots \quad 4.9.2$$

or
$$Y(k) = \frac{F(k)}{1-\sqrt{2\pi}\,\lambda K(k)} \quad \cdots\cdots \quad 4.9.3.$$

Taking the inverse Fourier transform

$$y(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{F(k)\, e^{ikx}}{1-\sqrt{2\pi}\,\lambda K(k)}\, dk \quad \cdots\cdots \quad 4.9.4$$

If the inverse F.T. can be done then the sol^n can be found explicitly, otherwise it must be left in the form of an integral.

Ex : Find the F.T of the f^n

$$g(x) = \begin{cases} 1 & \text{if } |x| \le a \\ 0 & \text{if } |x| > a \end{cases}$$

Hence find an explicit expression for the solution of the integral eq^n

$$y(x) = f(x) + \lambda \int_{-\infty}^{\infty} \frac{\sin(x-z)}{(x-z)}\, y(z)\, dz$$

Find the sol^n for the special case $f(x) = \frac{\sin x}{x}$

The F.T. of $g(x)$ is given by

$$G(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x)\, e^{-ikx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{-ikx}\, dx = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-ikx}}{(-ik)} \right]_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}(-ik)} \left[e^{-ika} - e^{ika} \right] = \frac{1}{\sqrt{2\pi}} \left(\frac{1}{-ik}\right)(-2i)\sin ka$$

$$= \sqrt{\frac{2}{\pi}}\, \frac{\sin ka}{k}$$

The kernel of the I.E is $k(x,z) = k(x-z) = \frac{\sin(x-z)}{(x-z)}$
using the above F.T. we find that the Fourier transform of the kernel $k(x-z)$ is

$$K(k) = \begin{cases} \sqrt{\pi/2} & \text{if } |k| \le 1 \\ 0 & \text{if } |k| > 1 \end{cases}$$

$$\frac{\sin(x-z)}{(x-z)} = \pi\,\delta(x-z)$$

F.T $= \sqrt{\pi/2}$

Integration over 'δ' f^n exists between limits ± 1.

$$\pi\,\delta_n(x-z) = \frac{\sin n(x-z)}{(x-z)} = \frac{1}{(2\pi)}\int_{-n}^{n} e^{i(x-z)t}\,dt.$$

Using 4.9.3., we find the F.T. of the soln to be

$$Y(k) = f(k)/(1-\pi\lambda) \quad \text{if } |k| \le 1$$

$$= f(k) \quad \text{if } |k| > 1$$

Taking inverse F.T we have

$$y(x) = \frac{1}{(1-\pi\lambda)}\frac{1}{\sqrt{2\pi}}\int_{-1}^{1} f(k)\,e^{ikx}\,dk$$

$$= \left[\frac{1}{(1-\pi\lambda)} - 1\right]\frac{1}{\sqrt{2\pi}}\int_{-1}^{1} f(k)\,e^{ikx}\,dk$$

$$+ \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} f(k)\,e^{ikx}\,dk$$

$$= f(x) + \left(\frac{\pi\lambda}{1-\pi\lambda}\right)\frac{1}{\sqrt{2\pi}}\int_{-1}^{1} f(k)\,e^{ikx}\,dk$$

we notice the $y(x) \sim \infty$ when $\lambda = 1/\pi$

when $f(x) = \sin x/x$

$$y(x) = \frac{\sin x}{x} + \left(\frac{\pi\lambda}{1-\pi\lambda}\right)\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}\sqrt{\frac{\pi}{2}}\,e^{ikx}\,dk$$

$$= \frac{\sin x}{x} + \left(\frac{\pi\lambda}{1-\pi\lambda}\right)\cdot\frac{1}{\sqrt{2\pi}}\sqrt{\frac{\pi}{2}}\left[\frac{e^{ikx}}{ix}\right]_{k=-1}^{k=1}$$

$$= \frac{\sin x}{x} + \left(\frac{\pi\lambda}{1-\pi\lambda}\right)\frac{1}{2}\frac{2i\sin x}{ix} = \frac{\sin x}{x} + \left(\frac{\pi\lambda}{1-\pi\lambda}\right)\frac{\sin x}{x}$$

$$= \frac{1}{(1-\pi\lambda)}\frac{\sin x}{x}$$

Note: If the I.E 4.9.1 has limits o to x making the eqn a Volterra type, then its soln can be found similarly by making ~~Laplace~~ Laplace transformation.

$$f(s) = \int_0^\infty f(t)e^{-st}dt \equiv \mathcal{L}[f(t)]$$

Ex: Consider the inhomogeneous Fredholm eqn.

$$y(x) = f(x) + \lambda\int_{-\infty}^{\infty} e^{-ixz} y(z)dz \quad \ldots\ldots\ldots 4.9.5$$

It has a kernel $k(x,z)e^{-ixz}$

The integral over z is just the F.T. of $y(z)$ apart from a constant. Hence

$$y(x) = f(x) + \lambda\sqrt{2\pi}\,\tilde{y}(x) \quad \ldots\ldots\ldots 4.9.6$$

where $\tilde{y}(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} y(z)\,e^{-ixz}dz \equiv$ F.T.

Taking the F.T. of 4.9.6 we have

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} y(x)\,e^{-ikx}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{-ikx}dx$$

$$+ \lambda\sqrt{2\pi}\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{y}(x)\,e^{-ikx}dx$$

or $\tilde{y}(k) = \tilde{f}(k) + \sqrt{2\pi}.\lambda.\tilde{g}(k)$ $\qquad$ $\tilde{g}(k)$ = F.T. of $\tilde{y}(x)$

changing variable k to x

$$\tilde{y}(x) = \tilde{f}(x) + \sqrt{2\pi}.\lambda.\tilde{g}(x) \quad \ldots\ldots\ldots 4.9.7$$

$$= \tilde{f}(x) + \sqrt{2\pi}\,\lambda\, y(-x)$$

substituting 4.9.7 in 4.9.6

$$y(x) = f(x) + \lambda\sqrt{2\pi}\left[\tilde{f}(x) + \sqrt{2\pi}.\lambda\,\tilde{g}(x)\right]$$

$\Rightarrow$ This can be farther simplified.

$$[\tilde{\tilde{y}}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, \tilde{y}(k)\, e^{-ikx}$$

$$= \frac{1}{2\pi} \iint_{-\infty}^{\infty} dx\, dz\, y(z)\, e^{-ix(k+z)}$$

$$= \int_{-\infty}^{\infty} y(z)\, \delta(k+z)\, dz = y(-k)$$

changing k to x

$$\tilde{\tilde{y}}(x) = y(-x)$$

Hence $y(x) = f(x) + \lambda\sqrt{2\pi}\left[\tilde{f}(x) + \sqrt{2\pi}\,\lambda\, y(-x)\right]$ 4.9.8

or $y(x) = f(x) + \sqrt{2\pi}\cdot\lambda\,\tilde{f}(x) + (2\pi)\lambda^2\, y(-x)$

$$= f(x) + \sqrt{2\pi}.\lambda\,\tilde{f}(x) + (2\pi)\lambda^2\left[f(-x) + \sqrt{2\pi}.\lambda.\tilde{f}(-x) + (2\pi)\lambda^2 y(x)\right]$$

Thus the solution to 4.9.5 is given by

$$y(x) = \frac{1}{[1-(2\pi)^2\lambda^4]}\left[f(x) + \sqrt{2\pi}\,\lambda\,\tilde{f}(x) + 2\pi\lambda^2 f(-x) + (2\pi)^{3/2}\lambda^3\,\tilde{f}(-x)\right] \quad \ldots \ldots 4.9.9$$

4.9.9 has an unique solution provided the denominator

$$1-(2\pi)^2\lambda^4 \neq 0$$

i.e. $\lambda \neq \pm \frac{1}{\sqrt{2\pi}}$ or $\pm i\frac{1}{\sqrt{2\pi}}$

These are the eigenvalues of the corresponding homogeneous eqn with $f(x) = 0$

Ex: solve $y(x) = \exp(-x^2/2) + \lambda\int_{-\infty}^{\infty} \exp(-ixz)\, y(z)\, dz$

where λ is a real constant. show that the solution is unique . unless λ has one of two particular values.

Here $f(x) = \exp(-x^2/2)$

$$f(-x) = f(x)$$

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\, e^{-ikx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{x^2}{2} + ikx\right)}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[\left(x^2 + 2ikx + (ik)^2\right) - (ik)^2\right]/2}\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x+ik)^2/2}\, dx\; e^{-k^2/2}$$

$$= e^{-k^2/2}\, \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x+ik)^2/2}\, dx.$$

$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x+ik)^2/2}\, dx \Rightarrow$ can be shown to be equal to unity using complex Integration. In fact this is Gaussian normalisation integration.

Thus $\tilde{f}(k) = e^{-k^2/2}$

changing $\tilde{f}(x) = e^{-x^2/2}$
$k \to x$

Thus the solⁿ is by eqⁿ 4.9.9

$$y(x) = \frac{1}{1-(2\pi)^2\lambda^4}\left[1+\sqrt{2\pi}\,\lambda + 2\pi\lambda^2 + (2\pi)^{3/2}\lambda^3\right]e^{-x^2/2} \quad \ldots\ldots 4.9.10$$

since λ is real by assumption, the solⁿ will be unique unless

$$\lambda = \pm\frac{1}{\sqrt{2\pi}}$$

at which $y(x) \longrightarrow \infty$

Existence of solutions at these points can be analysed.